第三届核结构与反应少体研讨会@惠州

Extracting hadronic interaction from the lattice QCD raw data

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Southeast University

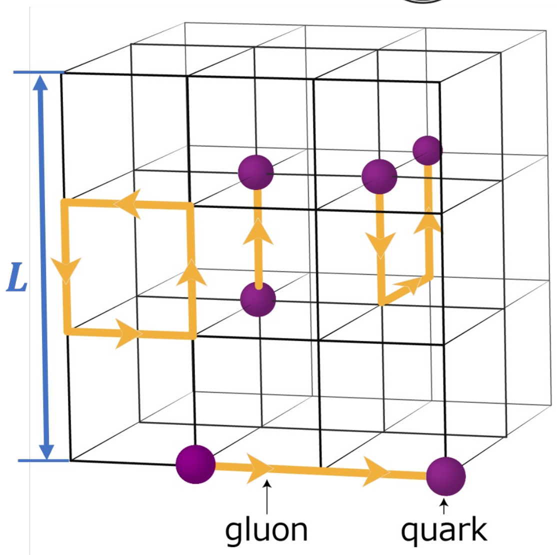
Jan. 21, 2026

Based on JHEP10(2021)051, PoS LATTICE2022 (2023) 201 and PRD109(2024), L071506 and papers in preparation

Together with V. Baru, E. Epelbaum, A. Filin, A.M. Gasparyan

From QCD to nuclei and neutron stars

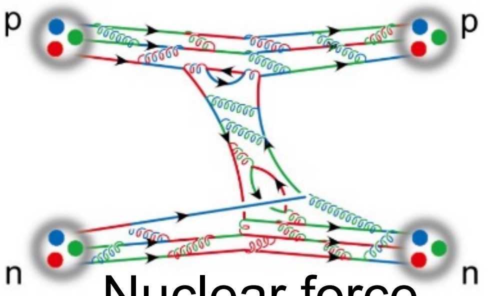
QCD: $\mathcal{L}_{QCD} = \sum_f \bar{q}_f (i\not{D} - \mathcal{M} q_f) - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu,a}$



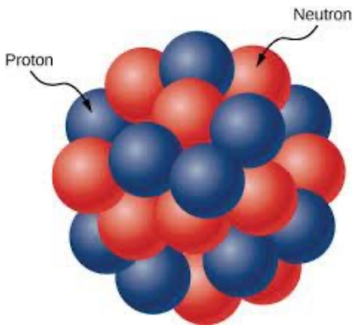
Lattice QCD

Phenomenological model

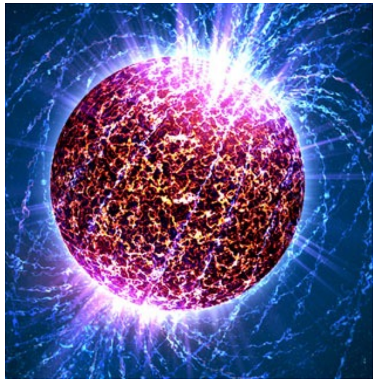
Chiral effective field theory



Nuclear force



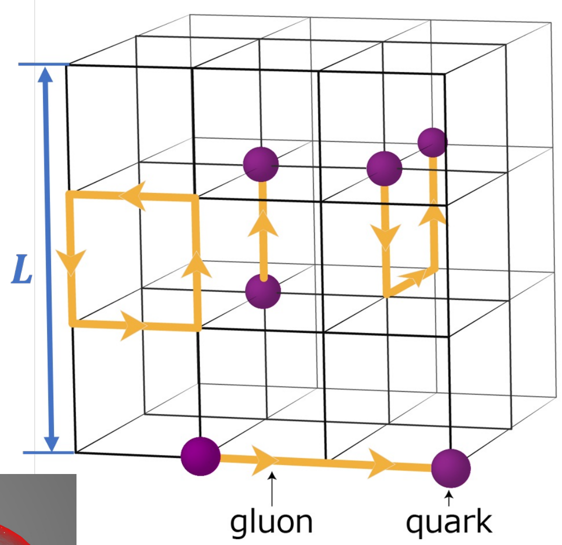
nucleus



neutron star

Hadronic molecule

QCD: $\mathcal{L}_{QCD} = \sum_f \bar{q}_f (i\not{D} - \mathcal{M} q_f) - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu,a}$



Lattice QCD

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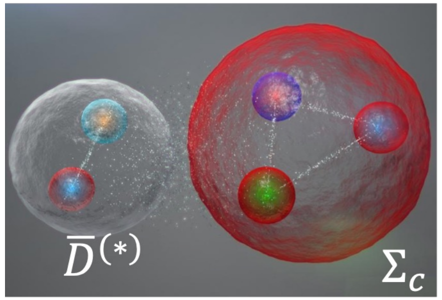
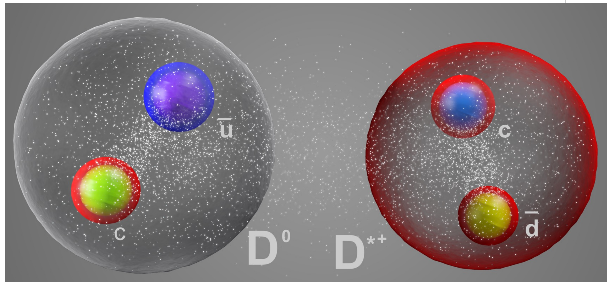




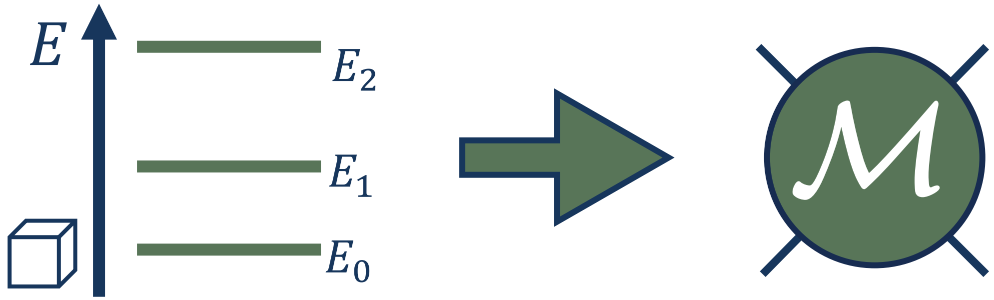
hadronic molecule

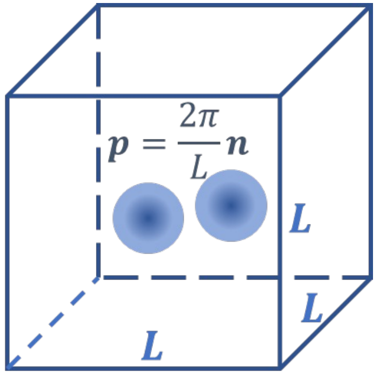
Energy level method

- Raw data: finite volume energy levels E^{FV}

Luscher:1990ux

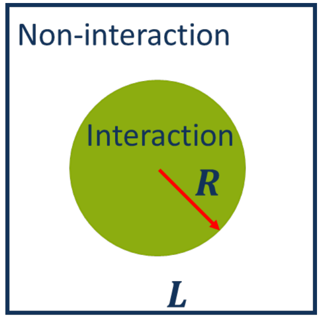
- Get observables: Lüscher's formula
  - ▶ $E^{FV} \sim \delta(E^{FV})$





Periodic boundary condition

  - ▶ AKA Lüscher quantization conditions



$\frac{L}{2} \gg R$

Asymptotic behavior: for $r > R$

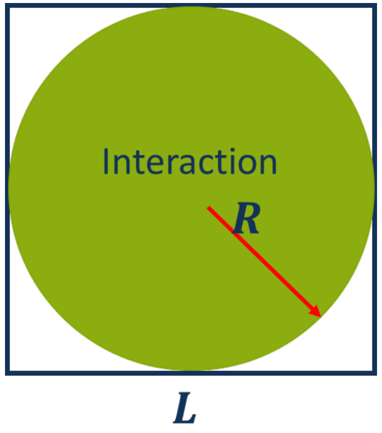
$$(\nabla^2 + k^2)\psi_k(r) = 0,$$

$$\psi_k(r) \sim \frac{e^{i\delta(k)}\sin[kr + \delta(k)]}{kr}.$$

Periodic Boundary condition:

$$\psi(r) = \psi(r + L)$$

$$p = \frac{2\pi}{L} n, \qquad n \in Z^3$$



Left-hand cut problem

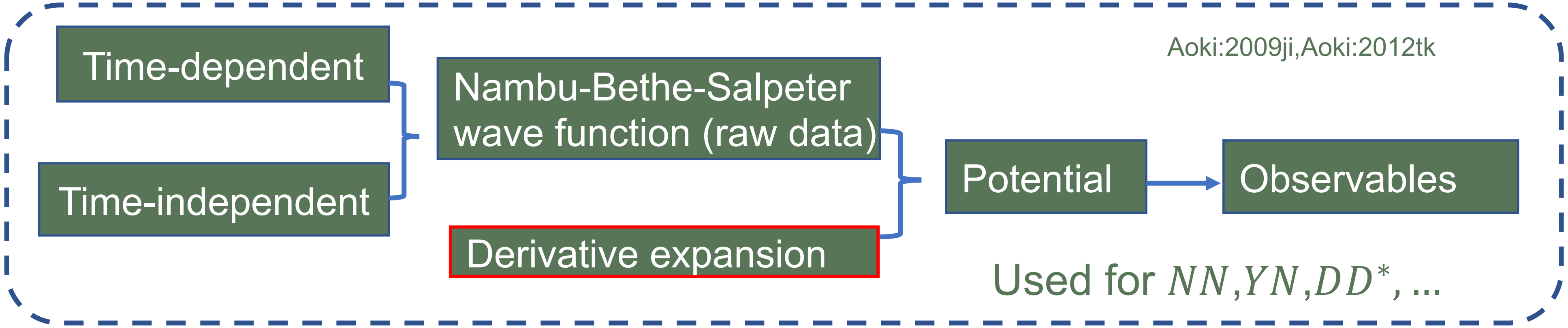
L. Meng et al, PRD109 (2024) 7, L071506;
Hansen et al. JHEP 06 (2024) 051;
Rishabh Bubna et al. JHEP 05 (2024) 168
…

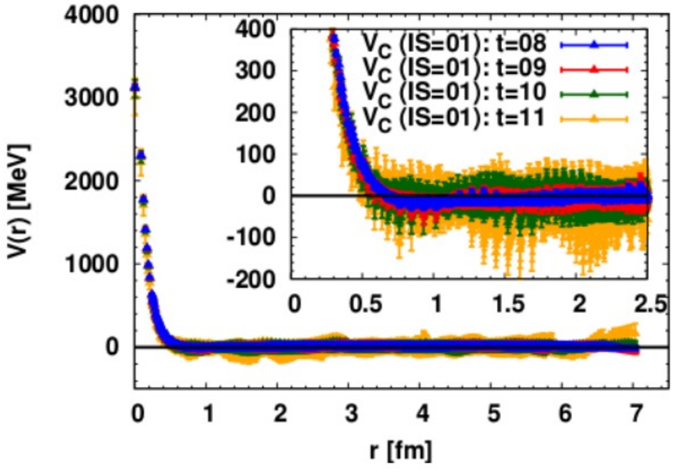
Long-range interaction and small box?

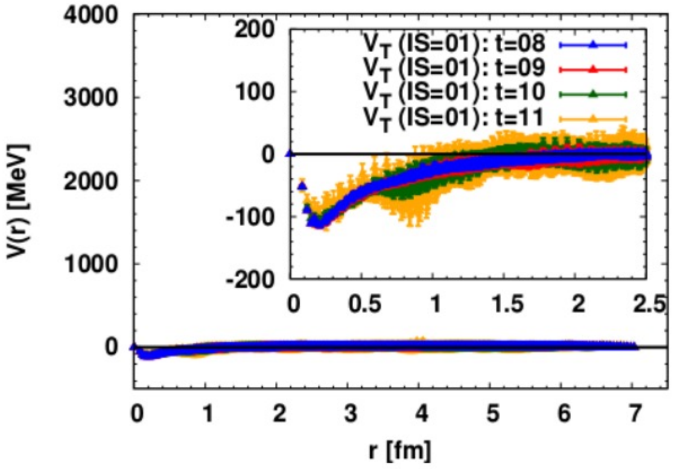
- Raw data: Nambu-Bethe-Salpeter wave function (NBS WFs)

Ishii:2006ec,Aoki:2009ji,Aoki:2012tk

AKA: HAL QCD (Hadrons to atomic nuclei from LQCD) method







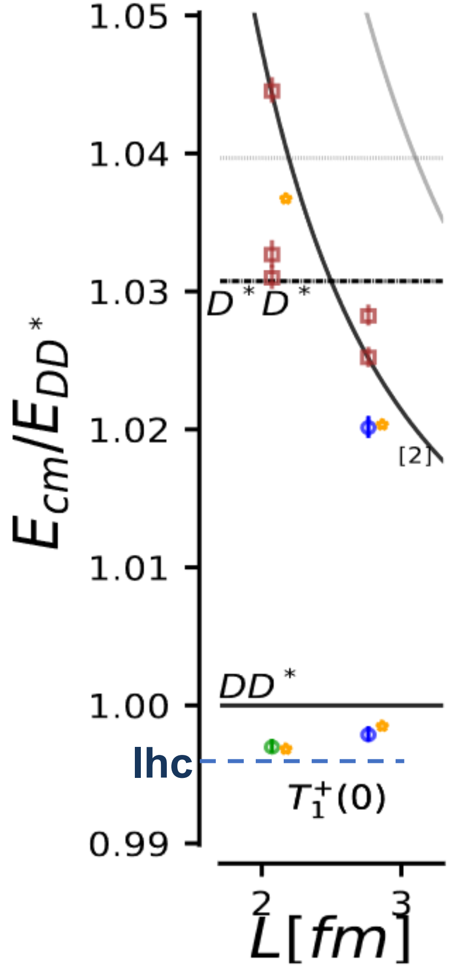
Contents

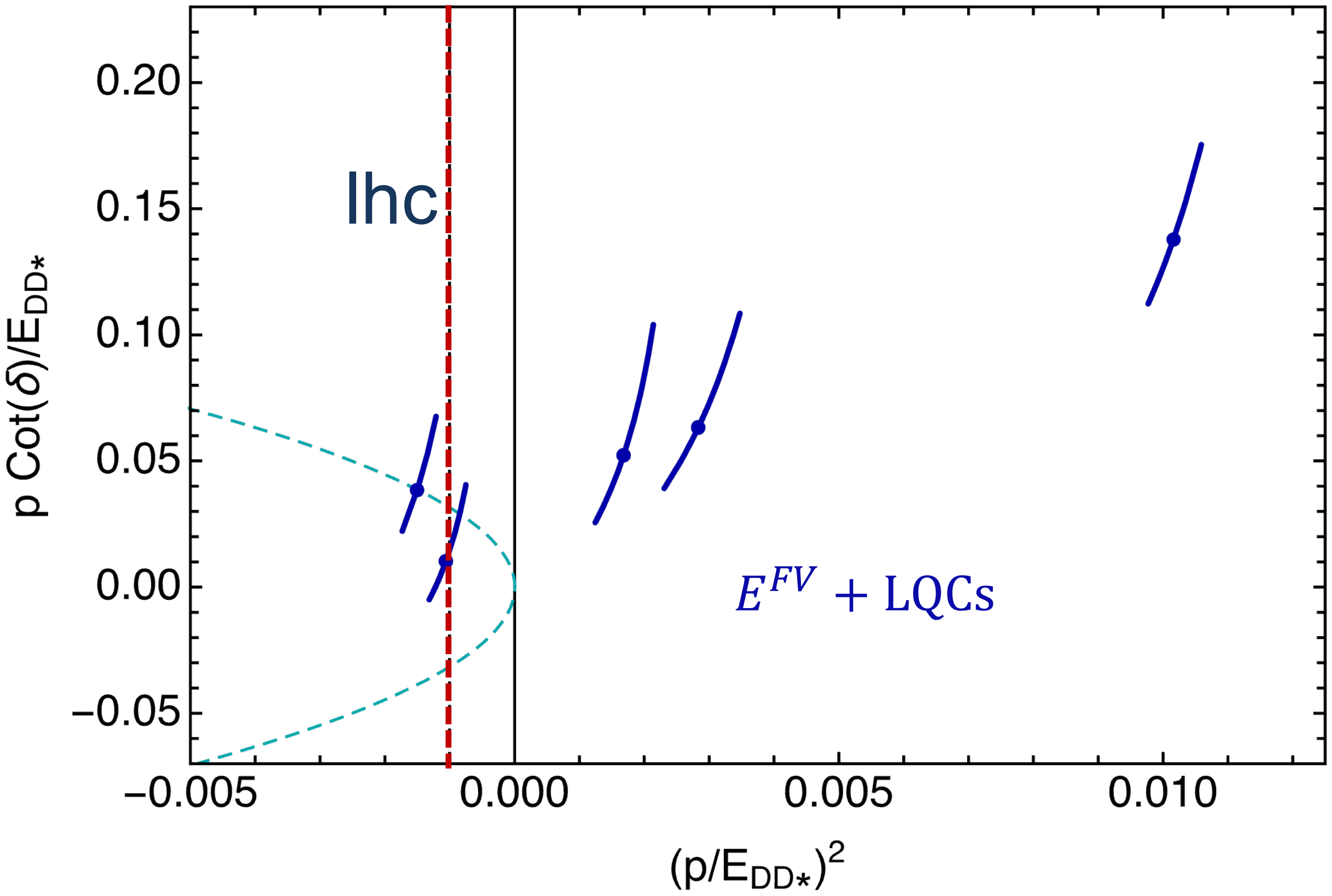
- **Part I:** Energy level method
  - **Left-hand cut problem**
  - Our solution: Hamiltonian method + Chiral EFT

14:00 – 15:30	Extracting ha data

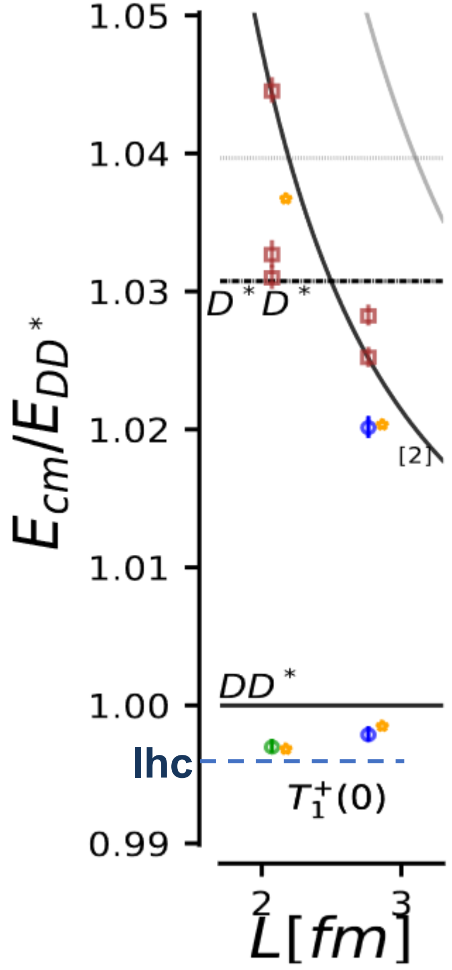
- **Part II:** Potential method
  - The general problem
  - Basics of scattering theory
  - Derivative expansion
  - Seperable expansion: EST method

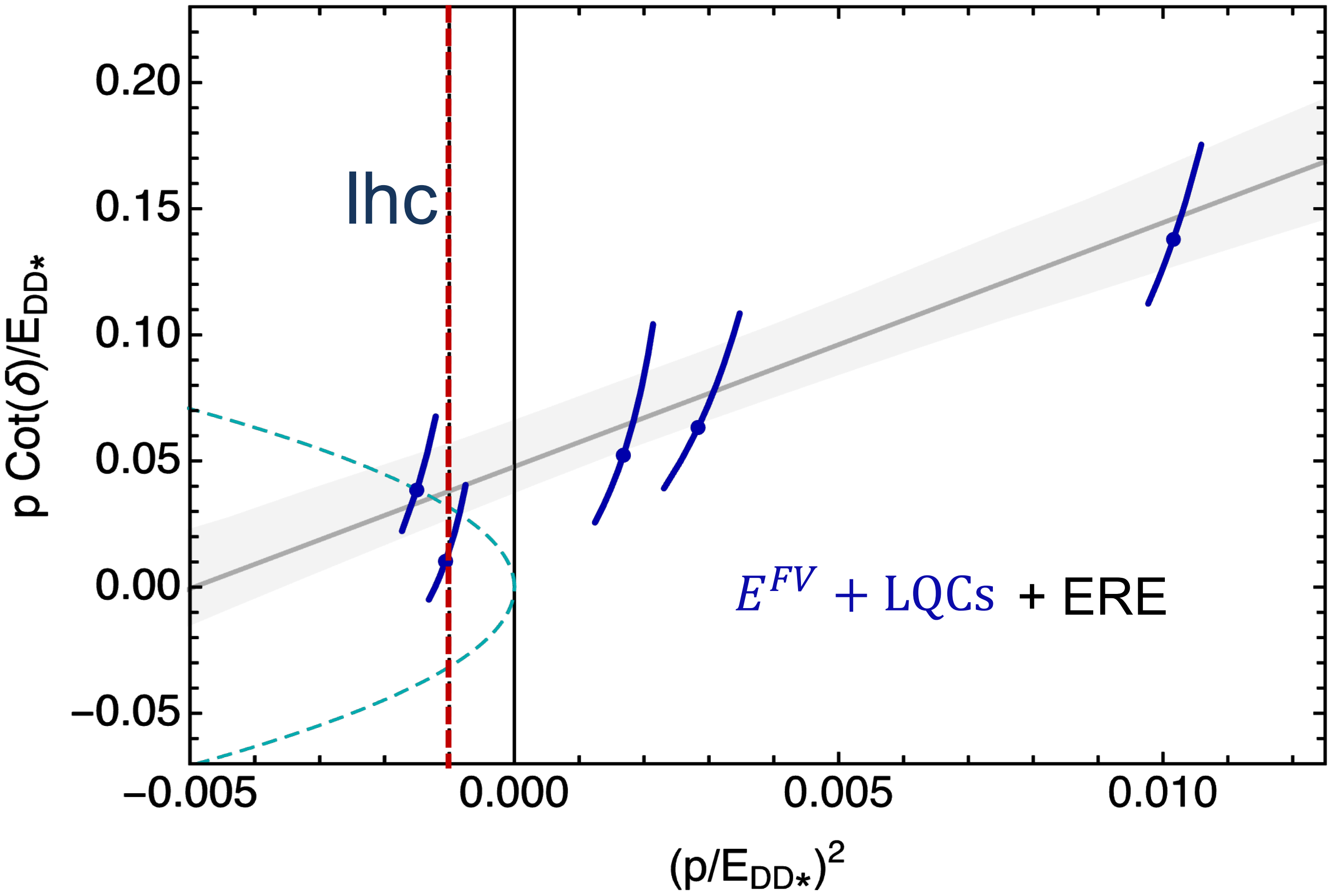
Left-hand cut problems





Left-hand cut problems





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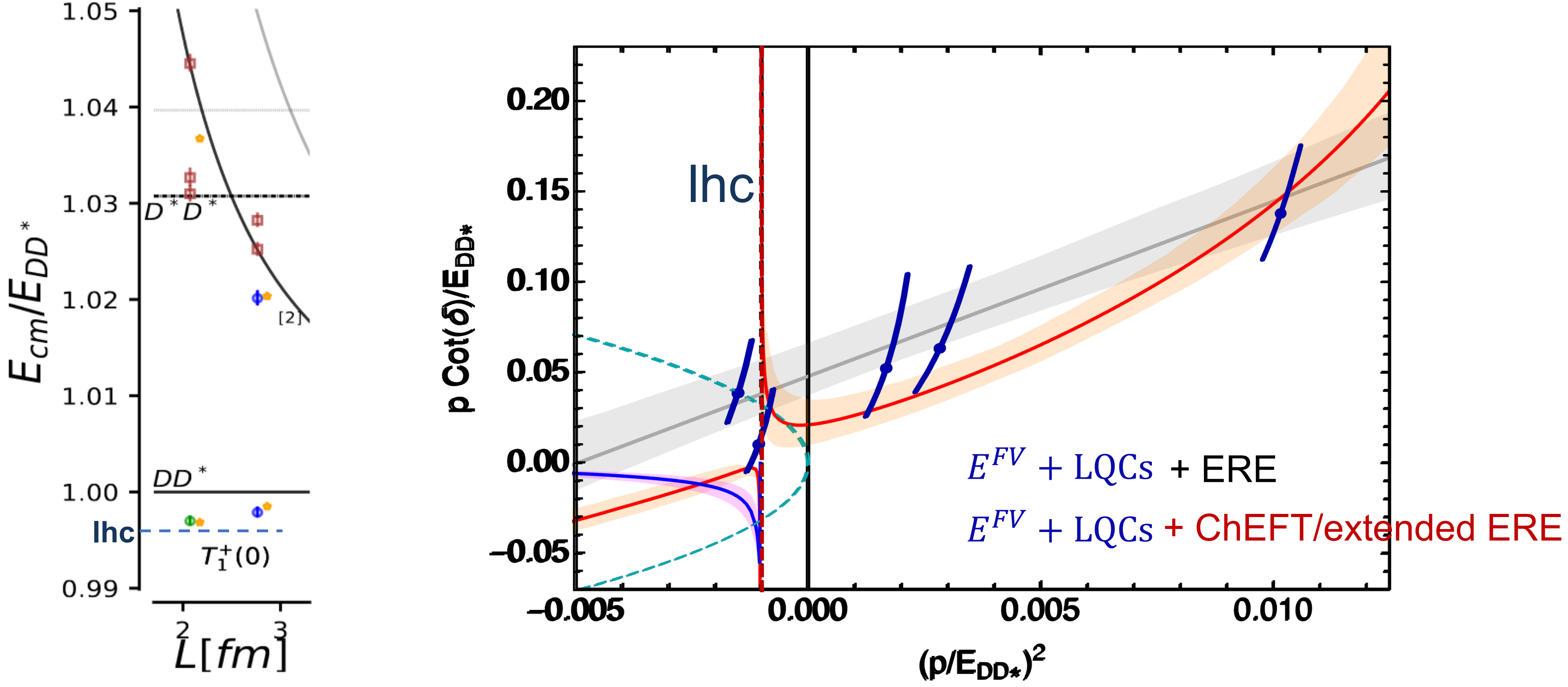


Left-hand cut problem

- Hamiltonian method

  J.-J. Wu, T.-S. H. Lee, et al, Phys. Rev. C **90**, 055206 (2014).
  L. Meng and E. Epelbaum, JHEP **10**, 051 (2021) (plane wave basis + irrps.)
  K. Yu, G.-J Wang, J.-J Wu et al, JHEP 04 (2025) 108 (helicity basis, EVP)
  Applications:
  L. Meng and E. Epelbaum, PoS **LATTICE2022**, 201 (2023). (NN)
  L. Meng, V. Baru, et al. Phys. Rev. D **109**, L071506 (2024). (T_{cc})
  L. Meng, E. Ortiz-Pacheco, et al, Phys. Rev. D **111**, 034509 (2025). ($DD^*, I = 1$)
  Sasa Prelovsek et al. Phys.Rev.D 112 (2025) 1, 014507 (DD^* with more operators)
  …

- Modified effective range expansion

  R. Bubna, H-W. Hammer, F. Müller, J-Y. Pang, A. Rusetsky and J-J. Wu, JHEP 05 (2024) 168

- Modified Lüscher quantization condition

  A. Raposo and M. Hansen, JHEP 08 (2024) 075

- Using three-particle formalism

  M. Hansen, F. Romero-López and S. Sharpe, JHEP *06 (2024) 051*

- N/D formalism

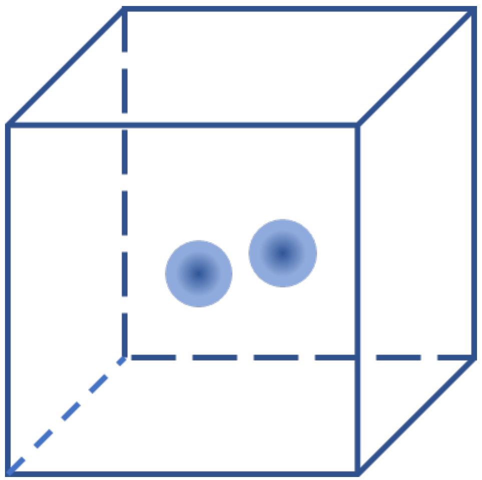
  S. M. Dawid, A. W. Jackura, and A. P. Szczepaniak, Phys. Lett. B **864**, 139442 (2025).

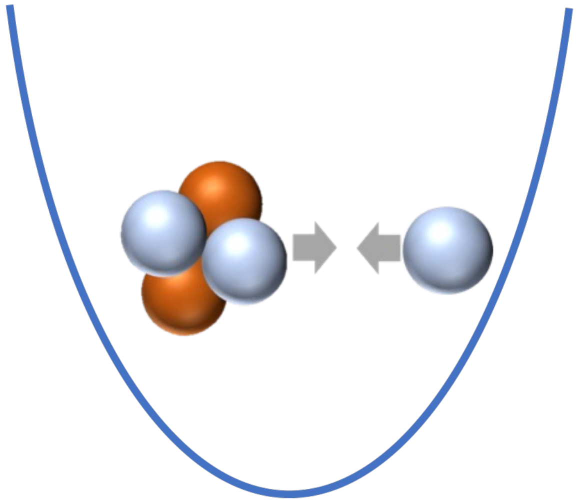
- Comparisons: S. M. Dawid e tal, JHEP **09**, 058 (2025). (only the formalism in the infinite volume)

Other traps

- Trapped in cubic box: Lüscher formula
- Harmoic trap: BERW formula

T. Busch, B.-G. Englert, K. Rzażewski, and M. Wilkens, Found. Phys. 28, 549 (1998).
A.Suzuki, Y. Liang, and R. K. Bhaduri, Phys. Rev. A **80**, 033601 (2009). (higher partial wave)
P. Guo and B. Long, J. Phys. G **49**, 055104 (2022). (coupled channel and different traps)
H. Zhang, D. Bai, Z. Wang, and Z. Ren, Phys. Lett. B **850**, 138490 (2024). (charged particles)
Y. Yang, E. Epelbaum, J. Meng, L. M, and P. Zhao, Phys. Rev. Lett. **135**, 172502 (2025). (n-alpha scattering)
…





Contents

Potential method

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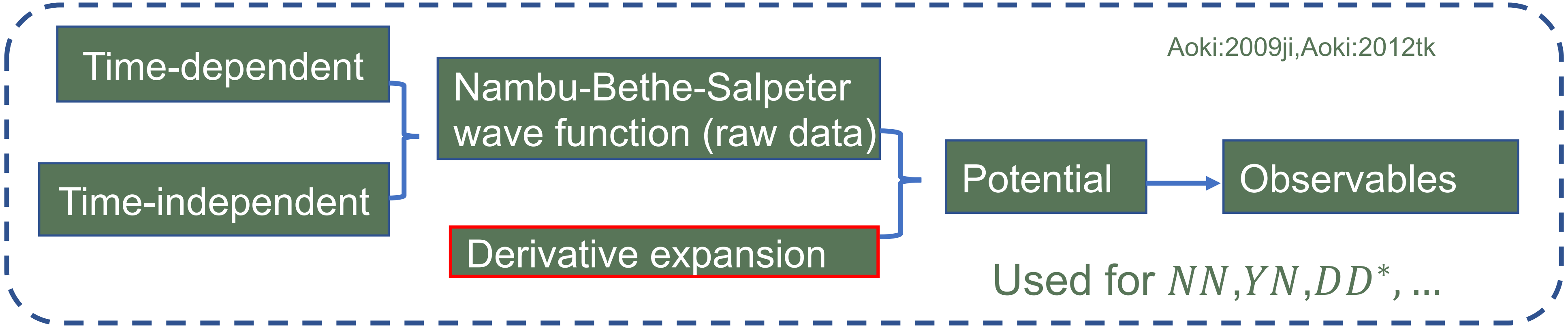

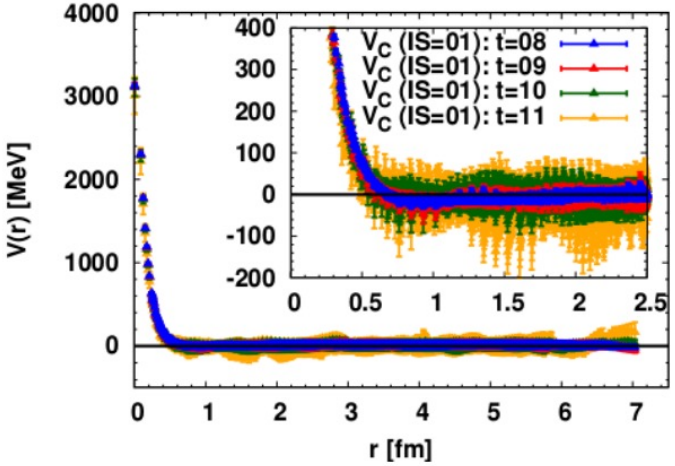

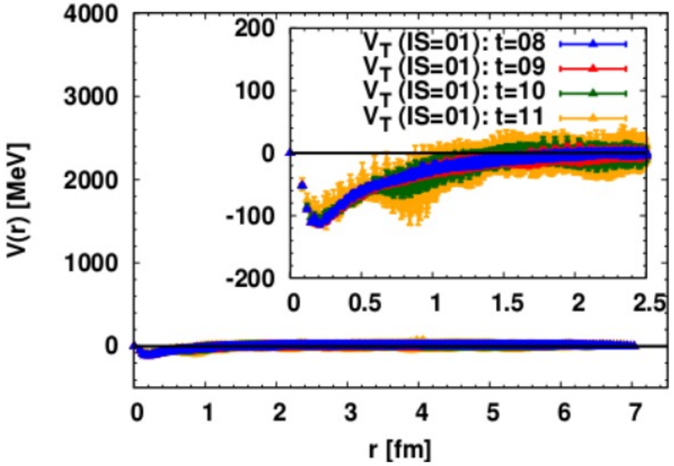

Potential from wave functions

- Asymptotic behavior of equal-time BS amplitude (BSWF, or NBS WFs) CP-PACS:2005gzm

$$\psi(\vec{x};\vec{k}) = e^{i\vec{k}\cdot\vec{x}} + \int \frac{d^3p}{(2\pi)^3} \frac{T(p;k)}{p^2-k^2-i\epsilon} e^{i\vec{p}\cdot\vec{x}}$$

  - ▸ $T(p;k)$: half-on-shell T-matrix
  - ▸ $\psi(\vec{x};\vec{k})$ satisfy the Lippmann-Schwinger eq. as the non-relativistic scattering wave function

- Time-independent HAL QCD (set $m=1$, 1D case as an example)

$$\int dr' V(r,r')\psi_{k_i}(r') = (\frac{d^2}{dr^2} + k_i^2)\psi_{k_i}(r)$$

- Time-dependent HAL QCD: correlation without ground state saturation Ishii:2012ssm

$$R(r,t) = \sum_n a_n \psi_{k_n}(r) e^{-(2\sqrt{m_N^2+k_n^2}-2m_N)t} \qquad \left(-\frac{\partial}{\partial t} + \frac{1}{4m_N}\frac{\partial^2}{\partial t^2}\right) R(r,t) = \left(\hat{H}_0 + \hat{V}\right) R(r,t)$$

- General problem: get potential $V(r,r')$ once $\{\Psi_{k_i}(r)\}$ or $\{R_i(r)\}$ are given

$$\int dr'\, V(r,r') R^{(i)}(r') = K^{(i)}(r)$$

$R^{(i)}(r)$ and $K^{(i)}(r)$ are known

To bind or not to bind

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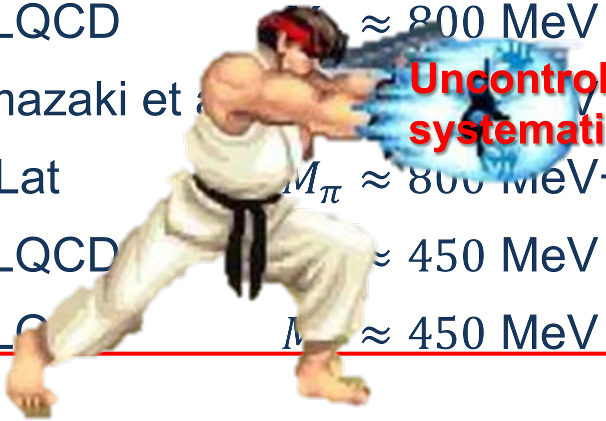
With (deeply) bound NN

1

2006	NPLQCD	First dynamical calculations
2011	NPLQCD	$M_\pi \approx 390$ MeV

2

2012	Yamazaki et al.	$M_\pi \approx 510$ MeV
2015	NPLQCD	≈ 800 MeV
2015	Yamazaki et a	
2015	CalLat	$M_\pi \approx$ 800 MeV+P,D,F waves
2015	NPLQCD	≈ 450 MeV
2020	NPL	≈ 450 MeV



Uncontrolled systematics

Without bound NN (or inconclusive)

2012	HAL QCD	$M_\pi \approx 710$ **MeV**
2012	HAL QCD	$M_\pi \approx 469 - 1171$ MeV

3

2019	"Mainz"	≈ 960 Me
2020	CoSMoN	$M_\pi \approx 714$ MeV
2021	NPLQCD	$M_\pi \approx 800$ MeV
2025	BaSc	$M_\pi \approx 714$ MeV

- However, we are observing a preponderance of evidence that the older methods with present statistics, are yielding qualitatively incorrect spectrum —
  I believe the old results are wrong (including those I was involved with)
  I believe the di-nucleon system unbinds at pion masses heavier than physical

Talk of A.Walker-Loud in lattice2023:https://indico.fnal.gov/event/57249/contributions/271301/

To bind or not to bind

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With (deeply) bound NN

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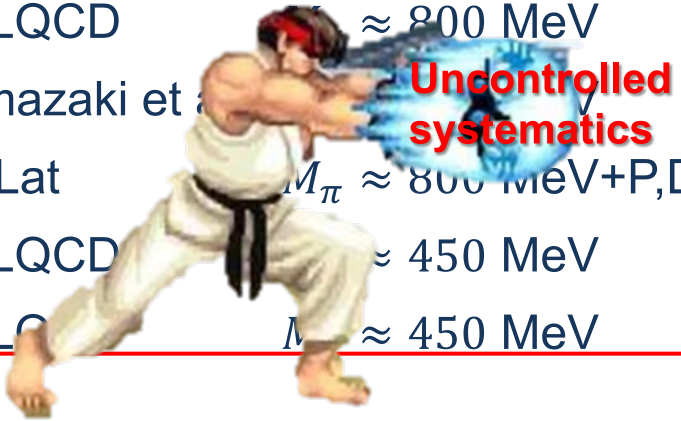

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	D	$M_\pi \approx 800$ MeV
		$M_\pi \approx 714$ MeV

Di-nucleons do not form bound states at heavy pion mass

BaSc Collaboration • John Bulava (Ruhr U., Bochum) Show All(19)

May 8, 2025

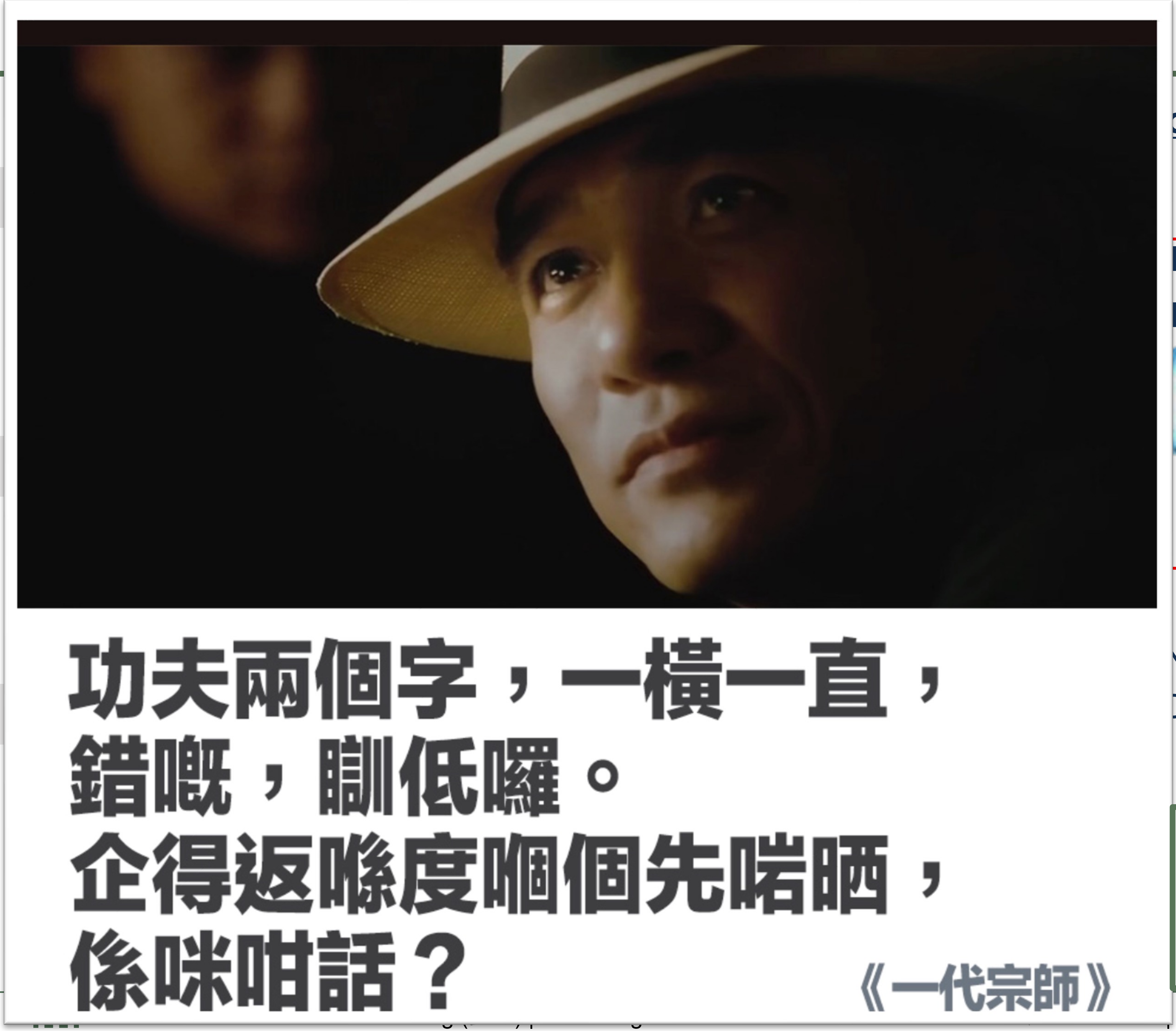
30 pages

e-Print: 2505.05547 [hep-lat]

Report number: LLNL-JRNL-2005660

View in: ADS Abstract Service

A benchmark of two methods



ound NN (or inconclusive)

D	$M_\pi \approx 710$ **MeV**
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	≈ 960 Me
N	$M_\pi \approx 714$ MeV
D	$M_\pi \approx 800$ MeV
	$M_\pi \approx 714$ MeV

Disclaimers:
- I am not the member of the HAL QCD group
- I will try my best to be fair

Contents

The only observavble: cross section

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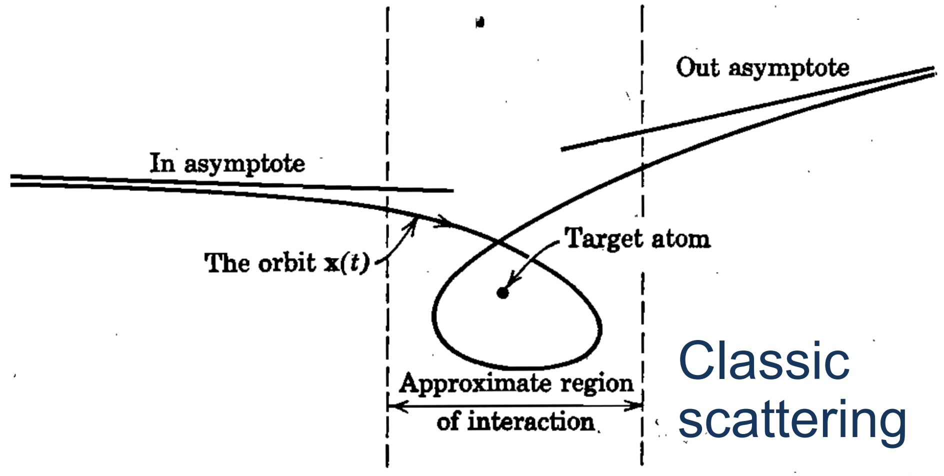
- Scattering theory: interactions occur over a finite region
  - ▶ Inner region (interacting region): $V(r) \neq 0 \ (r < R)$
  - ▶ outer region (asymptotic region): $V(r) = 0 \ (r' > R)$
- Asymptotic states (trajectories): particles are free long before and after the collision
- The observable: cross section



Classic scattering

On-shell scattering amplitude Phase shift Asymptotic wave function	⟷	Cross section

- Non-observable
  - ▶ Non-asymptotic behavior of ψ
  - ▶ Off-shell T-matrix
  - ▶ Potential

A related debat

E. Epelbaum, et al, **Can the strong interactions between hadrons be determined using femtoscopy**?, arXiv:2504.08631

Local potential vs Nonlocal potential

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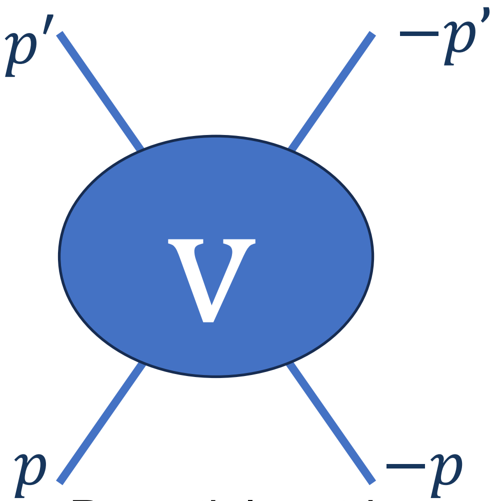
- General potential in coordinate and momentum space: (1D as an example)

$$\mathcal{V}(p',p)=\langle p'|V|p\rangle,\ \langle p|\hat{V}|\psi\rangle=(2\pi)^{-1}\int dp'\mathcal{V}(p,p')\psi(p')$$

$$V(r',r)=\langle r'|\hat{V}|r\rangle,\ \langle r|\hat{V}|\psi\rangle=\int dr' V(r,r')\psi(r')$$



Potential matrix element in C.M.F

- Representation with derivative operator

$$\begin{aligned}V(r',r)&=(2\pi)^{-2}\int dpdp'\mathcal{V}(p',p)e^{i(p'r'-pr)}\\&=(2\pi)^{-2}\int dkdq\bar{\mathcal{V}}(q,k)e^{i(ks+qR)}\\&=(2\pi)^{-2}\int dkdq\bar{\mathcal{V}}(q,-i\partial_s)e^{i(ks+qR)}\\&=\bar{V}(R,-i\partial_s)\delta(s)\end{aligned}$$

$$\begin{cases}R=\frac{r+r'}{2}, & q=p'-p,\\ s=r'-r, & k=\frac{p'+p}{2}.\end{cases}$$

$q = p' - p$ → exchanged

$k = \frac{p'+p}{2}$ → nonlocal

- No reason to reject nonlocal potentials: nonstatic, short-range, inelastic…

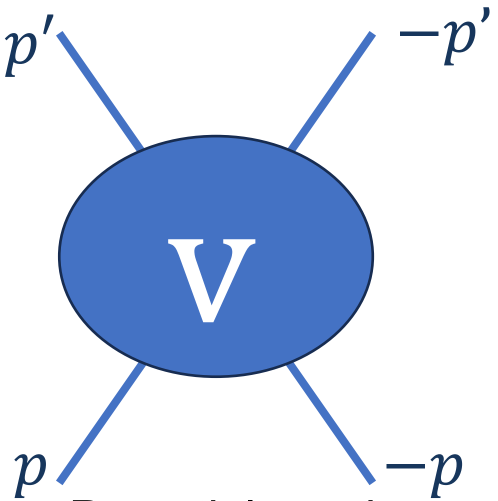
Tamagaki, R., & Watari, W. (1967). Potential model approach. *Prog. Theor. Phys. Suppl.*, *39*, 23–90

Local potential vs Nonlocal potential

- General potential in coordinate and momentum space: (1D as an example)

$$\mathcal{V}(p',p) = \langle p'|V|p\rangle,\ \langle p|\hat{V}|\psi\rangle = (2\pi)^{-1}\int dp'\mathcal{V}(p,p')\psi(p')$$

$$V(r',r) = \langle r'|\hat{V}|r\rangle,\ \langle r|\hat{V}|\psi\rangle = \int dr' V(r,r')\psi(r')$$



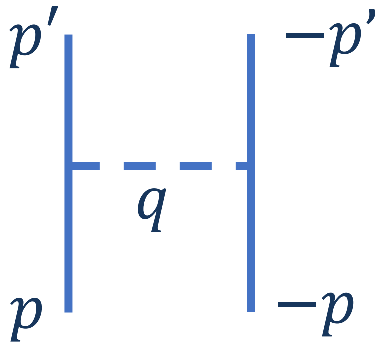
Potential matrix element in C.M.F

- Representation with derivative operator (local limit)

$$\begin{aligned} V(r',r) &= (2\pi)^{-2}\int dkdq\mathcal{V}(p',p)e^{i(ks+qR)} \\ &= (2\pi)^{-2}\int dkdq\bar{\mathcal{V}}(q)e^{i(ks+qR)} \\ &= \bar{V}(R)\delta(s) \\ &= \bar{V}(\frac{r+r'}{2})\delta(r'-r) \end{aligned}$$

$$\begin{cases} R = \frac{r+r'}{2}, & q = p'-p, \\ s = r'-r, & k = \frac{p'+p}{2}. \end{cases}$$

$$\langle r|\hat{V}|\psi\rangle = \int dr' V(r,r')\psi(r') = \bar{V}(r)\psi(r)$$

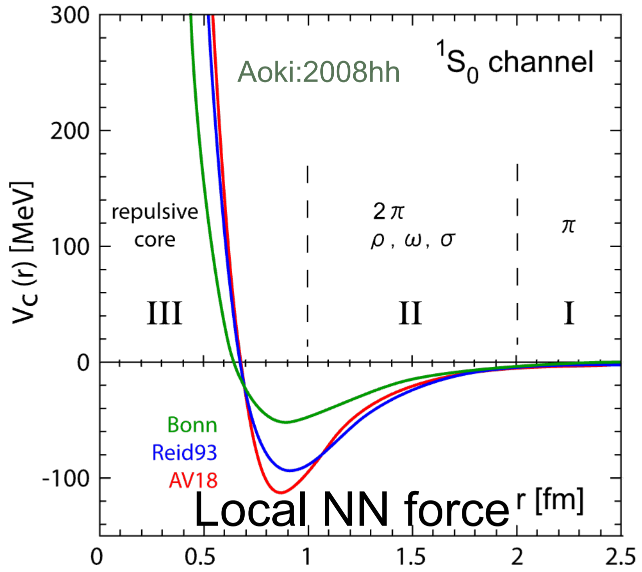


Static boson-exchange

- No reason to reject nonlocal potentials: nonstatic, short-range, inelastic…

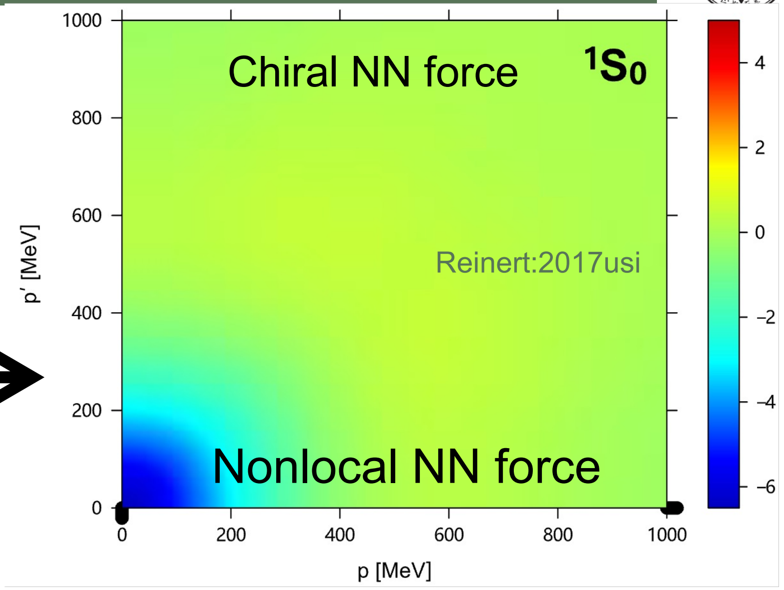
Tamagaki, R., & Watari, W. (1967). Potential model approach. *Prog. Theor. Phys. Suppl.*, *39*, 23–90

Unitary transformation



Phase shift equivalent

Unitary transformation



- Potential is not observable: cannot be determined uniquely by scattering experiments
  - Eg: high precision nuclear forces
- Observable-equivalent potentials are related by unitary trans. (UT) or field redefinition

$$H|\psi\rangle = E|\psi\rangle \Rightarrow UHU^{\dagger}U|\psi\rangle = EU|\psi\rangle \Rightarrow \tilde{H}|\tilde{\psi}\rangle = EU|\tilde{\psi}\rangle$$

  - Potential, Non-asymptotic ψ and off-shell T −matrix and are changed consistently
- UT can relate local potentials to nonlocal potentials
- Similarity renormalization group

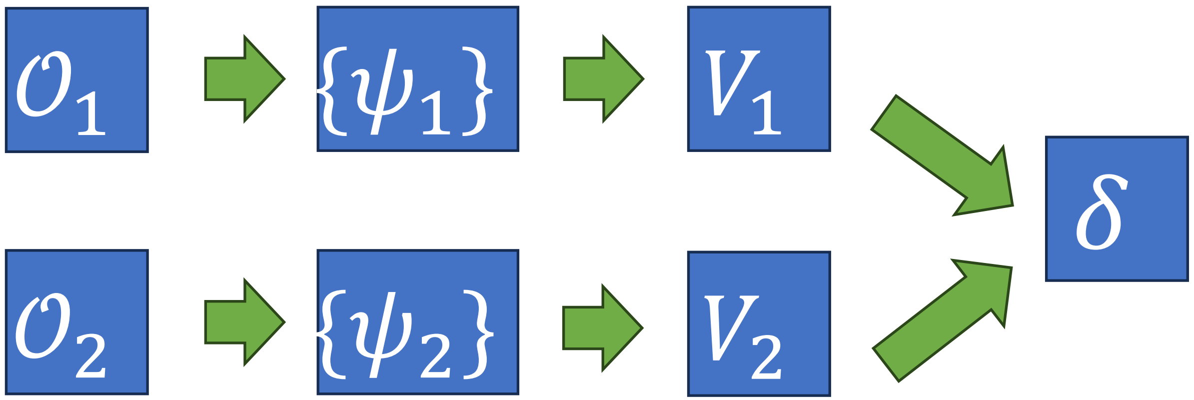
Ekstein:1960xkd

Query 1: potential is not observable

Query 1: Given the potential is not observable, is HAL QCD method still reasonable?

- In principle one may choose any composite operators with the same quantum numbers as the hadron to define the BS wave function
- Different operators give different BS wave functions and different hadron potentials
  - ▶ They are related by UT
  - ▶ We anticipate they lead to the same observables such as the δ and E_b

Aoki, S, et al, *PTEP*, *2012*, 01A105; S. Aoki's talk @Lattice 2019



Query 2: Additional information from WFs of interacting region?

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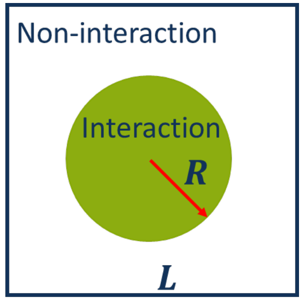
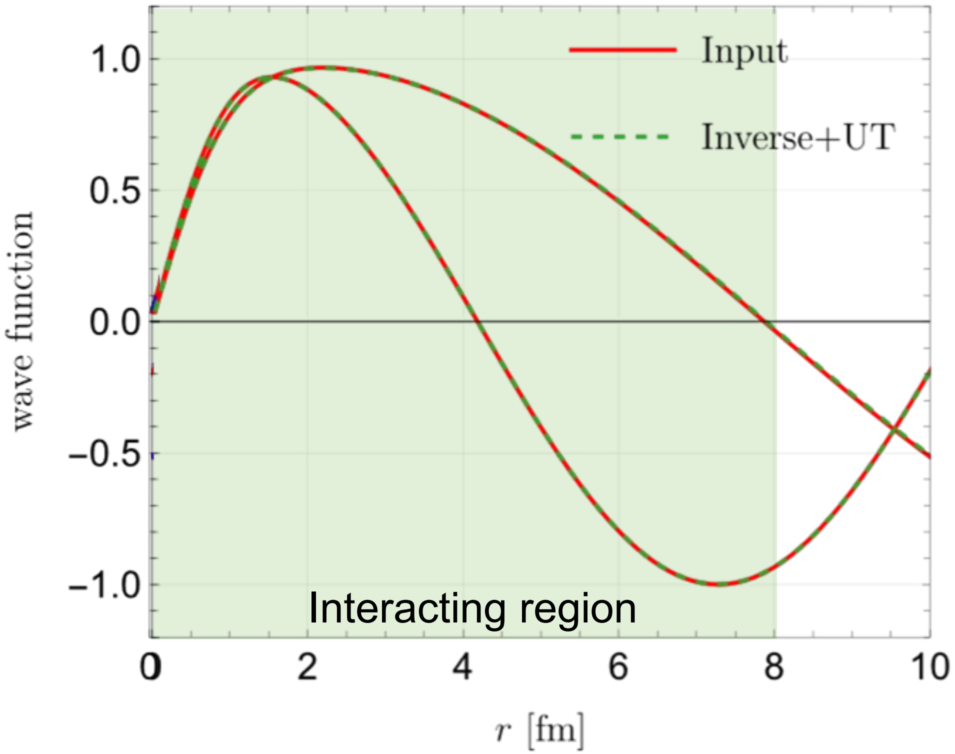
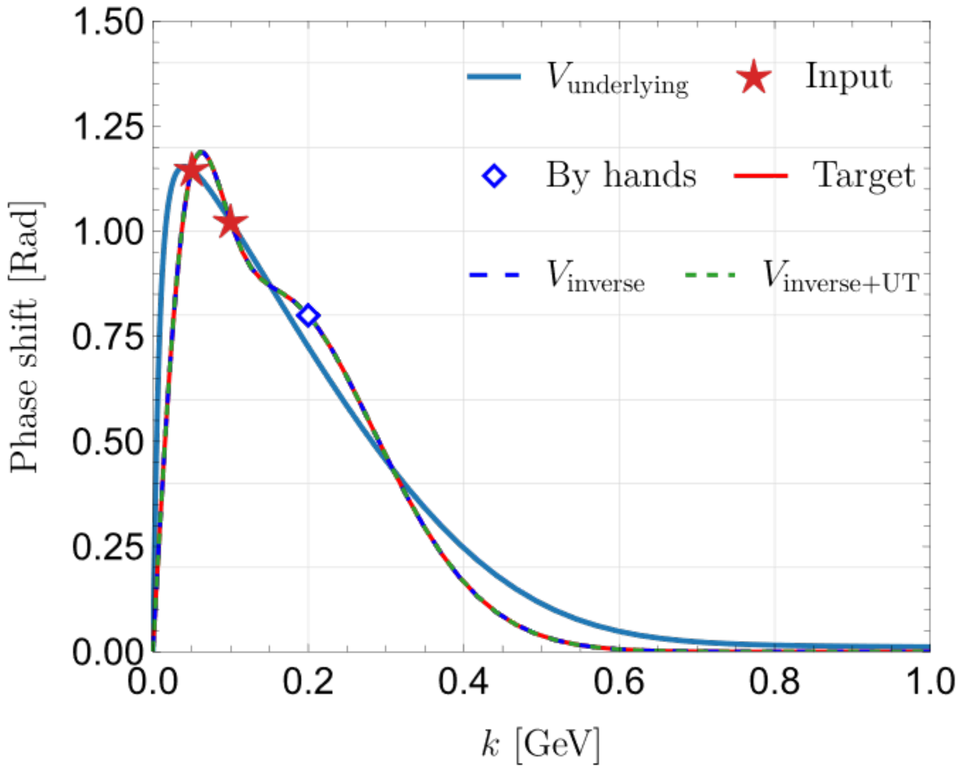
Query 2: Additional information from WFs of interacting region?

- Asymptotic region WF: several $\delta(k_i)$, similar to Luscher
- Interacting region WFs $\Rightarrow$ Potential $\Rightarrow$ $\delta(k)$
  - ► scheme-dependent

Yamazaki, T., & Kuramashi, Y. PRD*96*(2017), 114511.
And subsequent reply and comment
PRD98(2018), 038501; *PRD98*(2018), 0385012





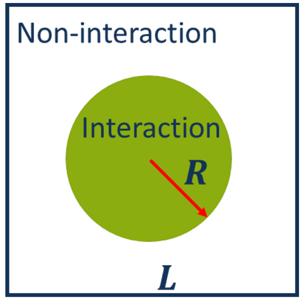


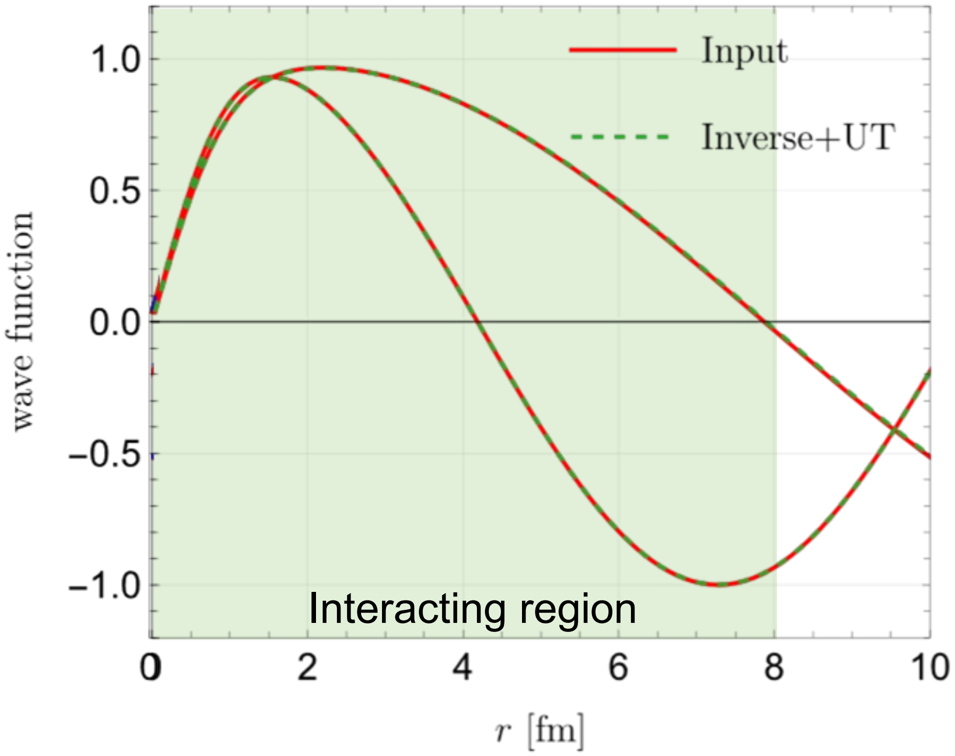
- To quantifying the uncertaities of phase shift other than $\delta(k_i)$: different parameterization of V

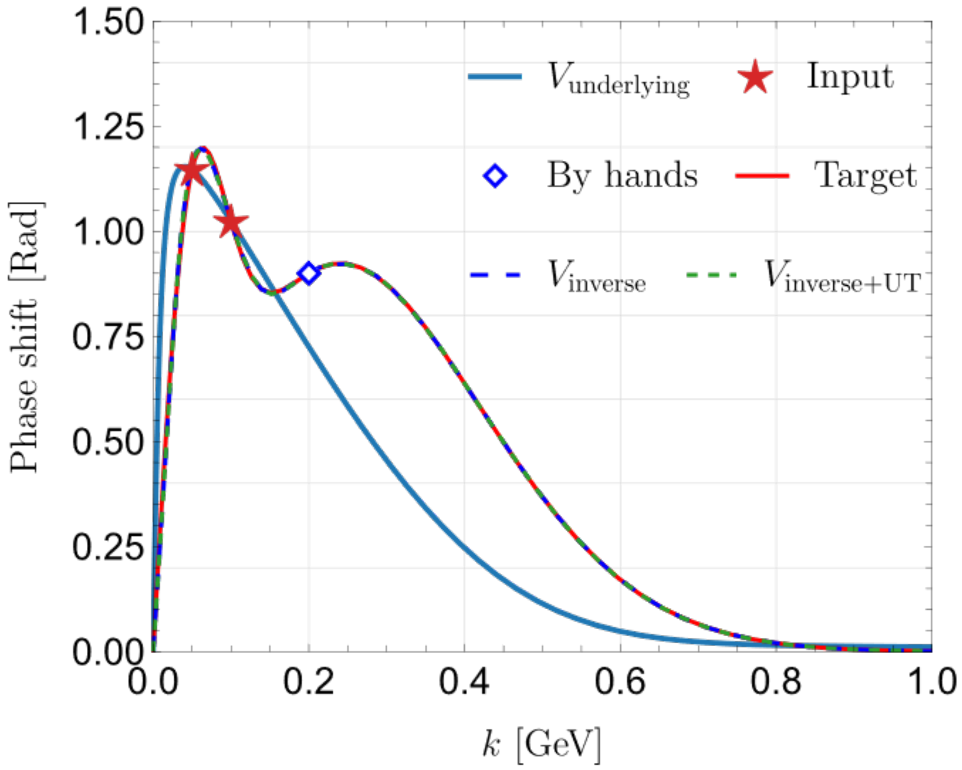
Query 2: Additional information from WFs of interacting region?

- Asymptotic region WF: several $\delta(k_i)$, similar to Luscher
- Interacting region WFs ⇒ Potential ⇒ $\delta(k)$
  - ► scheme-dependent

Yamazaki, T., & Kuramashi, Y. PRD*96*(2017), 114511.
And subsequent reply and comment
PRD98(2018), 038501; *PRD98*(2018), 0385012







- To quantifying the uncertaities of phase shift other than $\delta(k_i)$: different parameterization of V

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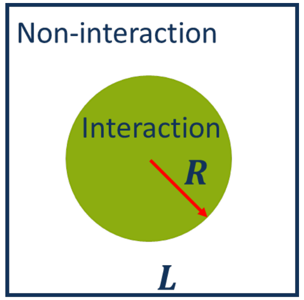

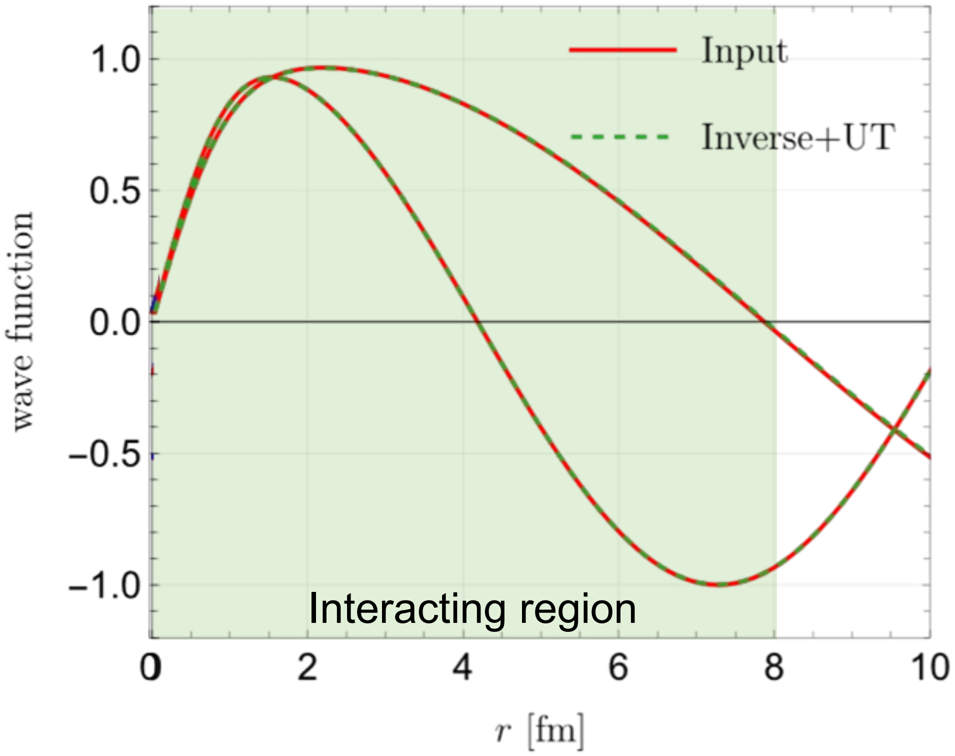

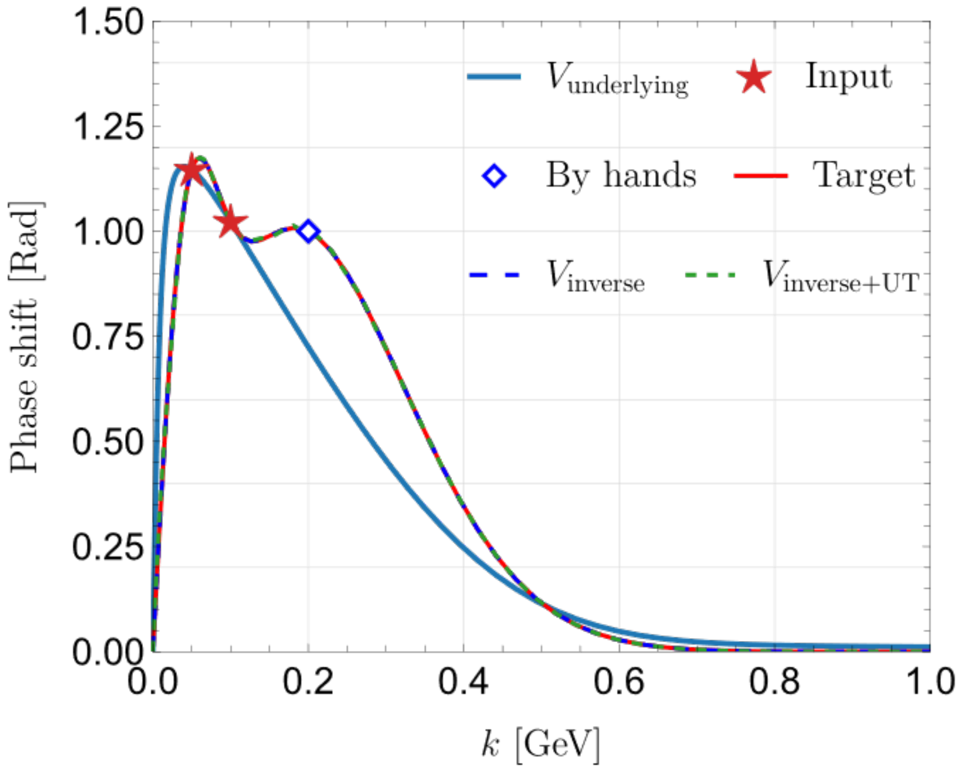

- To quantifying the uncertaities of phase shift other than $\delta(k_i)$: different parameterization of V

Contents

Derivative expansion

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- Derivative expansion

$$V(r,r') = V_0(r)\delta(r-r') + V_1(r)\delta(r-r')\frac{d^2}{dr'^2} + V_2(r)\delta(r-r')\frac{d^4}{dr'^4} + \dots$$

- LO

$$V_0(r)R^{(1)}(\vec{r}) = K^{(1)}(\vec{r}) \Rightarrow V_0(r) = \frac{K^{(1)}(\vec{r})}{R^{(1)}(\vec{r})}$$

- NLO

$$\begin{pmatrix} R^{(1)}(r) & \frac{d^2}{dr^2}R^{(1)}(r) \\ R^{(2)}(r) & \frac{d^2}{dr^2}R^{(2)}(r) \end{pmatrix} \begin{pmatrix} V_0(r) \\ V_1(r) \end{pmatrix} = \begin{pmatrix} K^{(1)}(r) \\ K^{(2)}(r) \end{pmatrix}$$

Aoki, S., & Yazaki, K. (2022). *PTEP*, *2022*(3), 033B04.

- Advantages:
  - ► Efficient for long-range potential: pionic dynamics, chiral symmetry
    E.g two-pion tail of DD^* interaction Y. Lyu *et al.*, Phys. Rev. Lett. **131**, 161901 (2023).
  - ► Local potential: easy for many-body calculation
- Several possible problems beyond the LO
  - ► Hermitian problem
  - ► Singular potential
  - ► Unconvergent potential⇒unconergent three-body observables

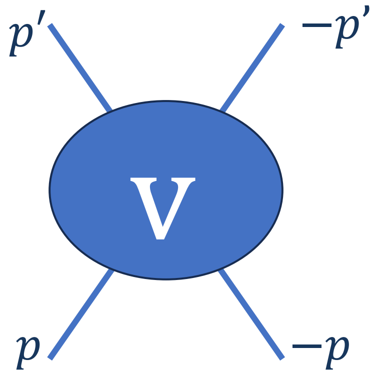
Hermitian

- Derivative expansion of HAL QCD group

$$V(r,r') = V_0(r)\delta(r-r') + V_1(r)\delta(r-r')\frac{d^2}{dr'^2} + V_2(r)\delta(r-r')\frac{d^4}{dr'^4} + \ldots$$

  - ▶ Non-Hermitian: all the differential operators acting to the right
  - ▶ $\mathrm{d}^2 R(r)/dr^2$ can be calculated directly ⇒ algebra eq.
  - ▶ Equivalent expansion in momentum space

$$V(p',p) = \tilde{V}(q,p) = \tilde{V}_0(q) + \tilde{V}_1(q)p + \frac{1}{2}\tilde{V}_2(q)p^2 + \ldots$$



$$q = p' - p,$$
$$k = \frac{p'+p}{2}.$$

- Optional hermitian Derivative expansion:

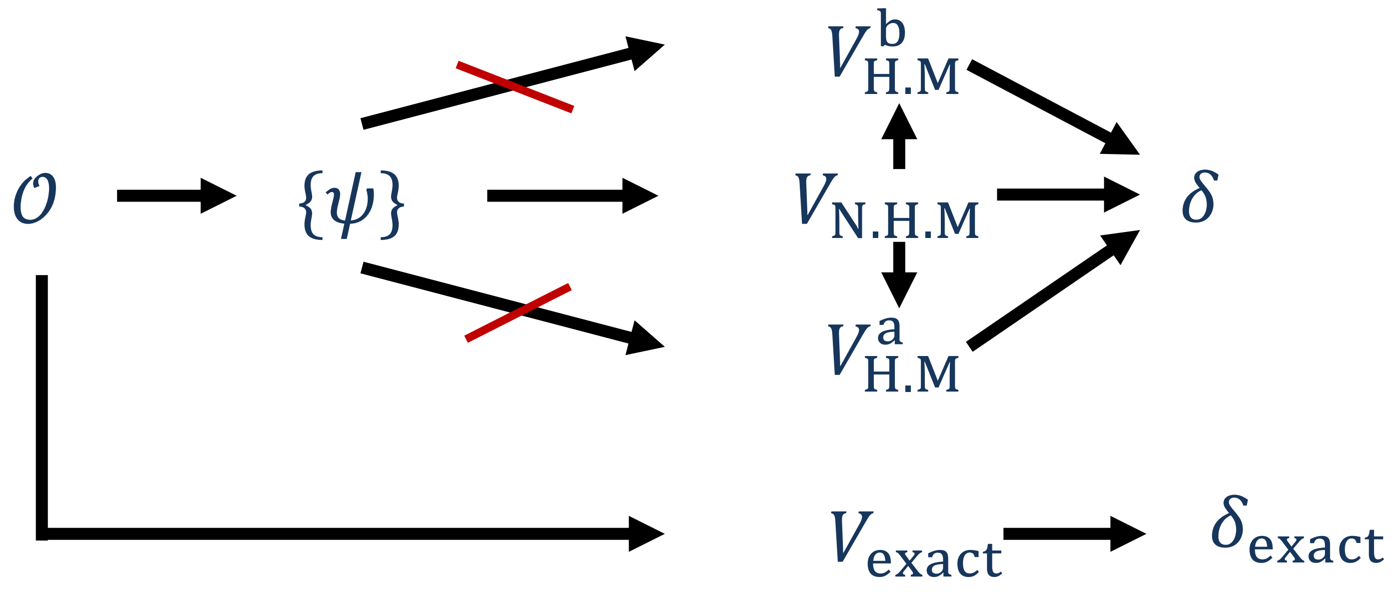
$$V(p',p) = \bar{V}(q,k) = \bar{V}_0(q) + \bar{V}_1(q)k + \frac{1}{2}\bar{V}_2(q)k^2 + \ldots$$

  - ▶ Have to solve differenttial eq.

$$V_1(r)\frac{\overrightarrow{d}^2}{dr} + \frac{\overleftarrow{d}^2}{dr}V_1(r) = V_1''(r) + 2V_1'(r)\frac{\overrightarrow{d}}{dr} + 2V_1(r)\frac{\overrightarrow{d}^2}{dr^2}$$

Hermitization?

- Hermitization of the non-Hermitian potential
  - Scheme-dependent
  - Inconsistent 3-body observable
  - Inconsistent with the exact observable



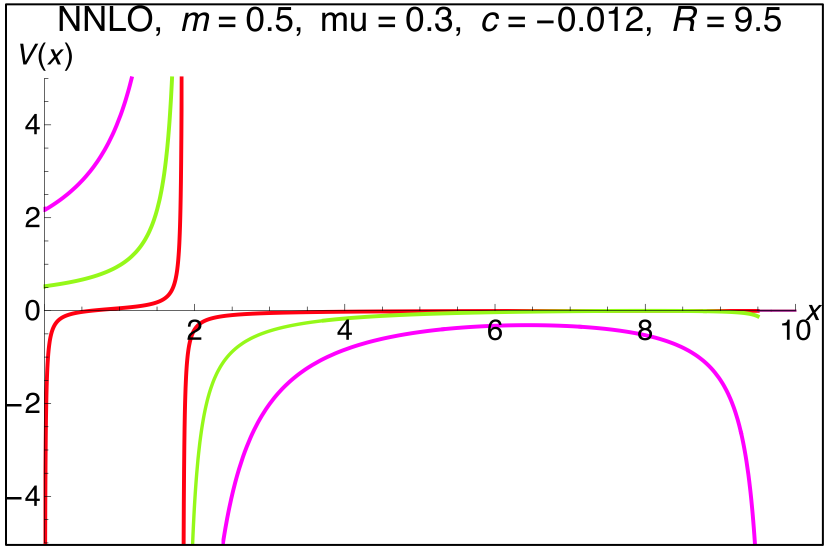
Aoki, S., Iritani, T., & Yazaki, K. (2020). *PTEP*, *2020*(2), 023B03.

Singular potential

- Singular potential
- NLO derivative expansion

$$\begin{pmatrix} R^{(1)}(r) & \frac{d^2}{dr^2}R^{(1)}(r) \\ R^{(2)}(r) & \frac{d^2}{dr^2}R^{(2)}(r) \end{pmatrix} \begin{pmatrix} V_0(r) \\ V_1(r) \end{pmatrix} = \begin{pmatrix} K^{(1)}(r) \\ K^{(2)}(r) \end{pmatrix}$$

- singular at the zero of det of the coefficient matrix
- An example from toy model
  - ▶ In simulation, it is challenging to handle the singularity
  - ▶ Wave functions are obtained at discrete point.



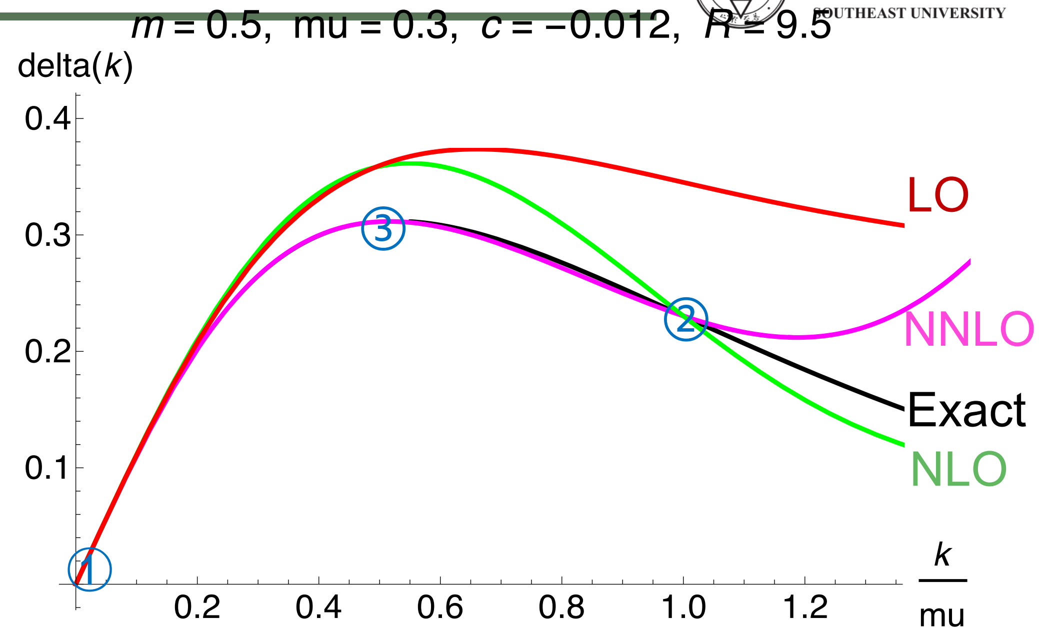
Aoki, S., & Yazaki, K. (2022). *PTEP*, *2022*(3), 033B04.

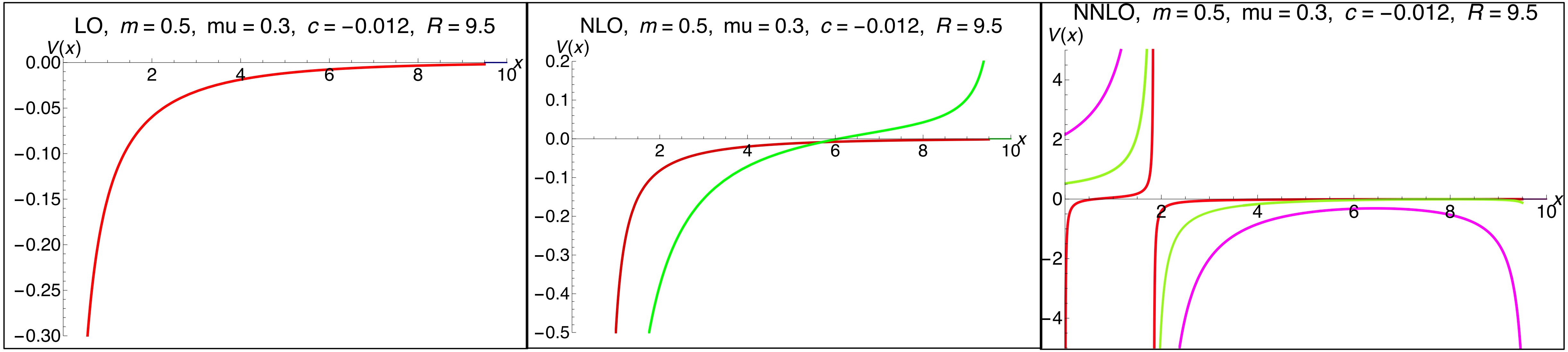
Convergence

- Derivative exansion of the separable interaciton
- Three wave functions ①②③ as input in order
- The 2-body phase shift is improved with order

However,

- The potential shape does not convergent
- The 3-body observables?





Aoki, S., & Yazaki, K. (2022). *PTEP*, *2022*(3), 033B04.

Contents

Separable representation

- The problem: $V|R^{(i)}\rangle = |K^{(i)}\rangle$

$$K^{(i)} = (E - \hat{H}_0)R^{(i)}$$

Ernst, D. J., Shakin, C. M., & Thaler, R. M. (1973). *Phys. Rev. C*, *8*, 46–52.

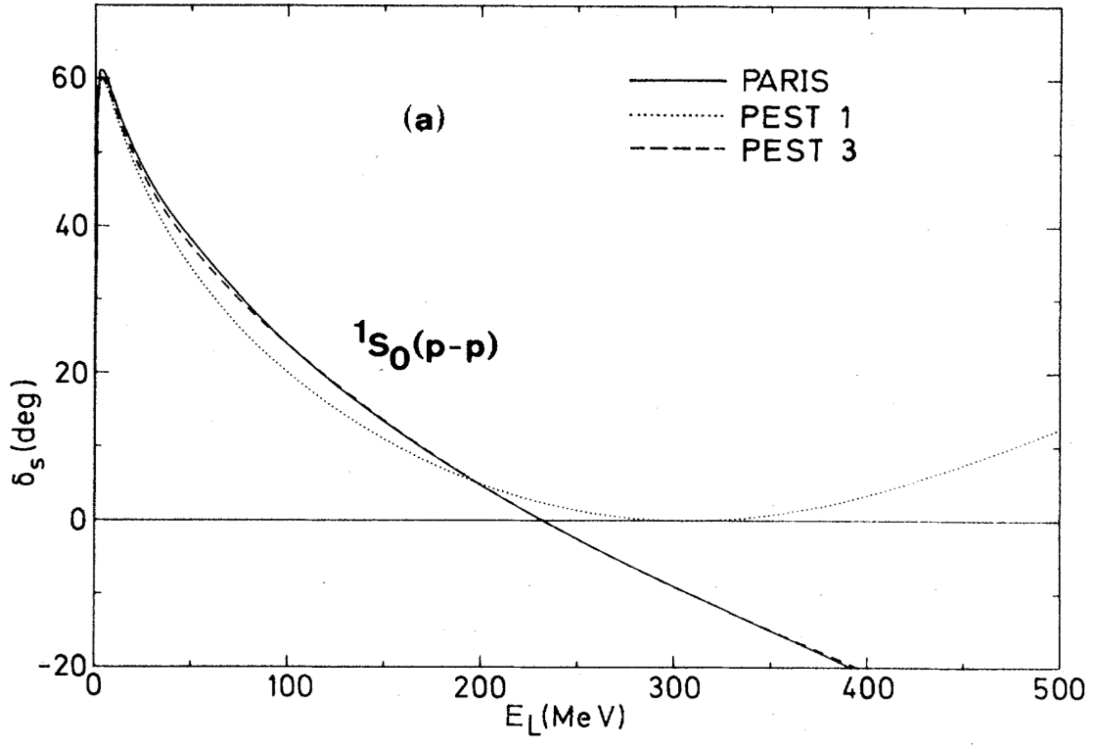
- Ernst-Shakin-Thaler (EST) method:

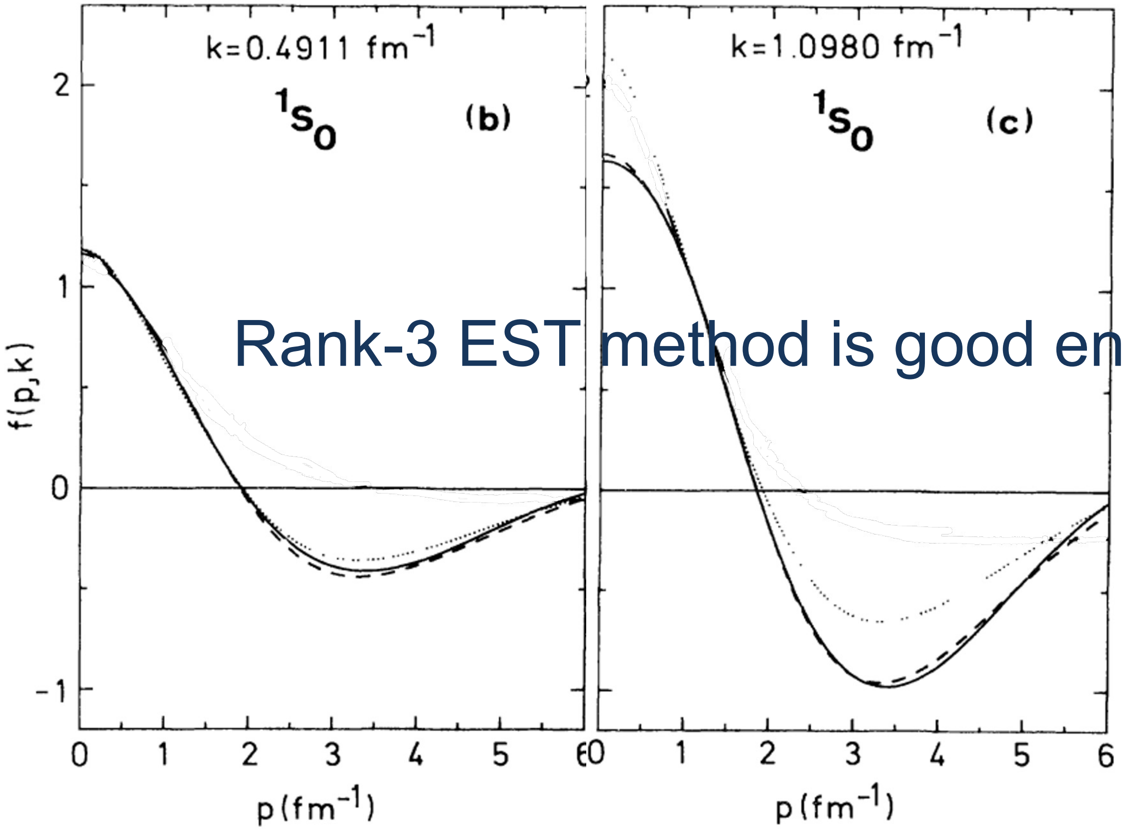
$$\hat{V} = \sum_{mn} |K^{(m)}\rangle \Lambda_{mn} \langle K^{(n)}|, \quad \sum_{n} \Lambda_{mn} \langle K^{(n)}|R^{(i)}\rangle = \delta_{mi}$$

- ▶ Hermitian potential
- ▶ No singularity
- ▶ On-shell and off-shell equivalent: three-body observable convergent

Separable representation

- Application: on-shell and off-shell equivalent separable potentials of Paris potentials
  - ► Purpose: to solve the 3-body Fadeev function
  - ► Convergent 3-body observables
  - ► High efficiency





Rank-3 EST method is good enough

Haidenbauer, J., & Plessas, W. (1984). *Phys. Rev. C*, *30*, 1822–1839.

Two underlying potentials

- Separable potential Aoki:2021ahj

$$V(\boldsymbol{r},\boldsymbol{r}') = \omega \frac{e^{-\mu r}}{r}\frac{e^{-\mu r'}}{r'}$$

- LO chiral nuclear force Reinert:2017usi

$$V_{ctc}(\boldsymbol{p},\boldsymbol{p}') = C e^{-\frac{p^2+p'^2}{\Lambda^2}}, \quad V_{ope}(\boldsymbol{q}) = -\frac{g_A}{4F_\pi^2}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{q}\boldsymbol{\sigma}_2\cdot\boldsymbol{q}}{\boldsymbol{q}^2+m_\pi^2} + C_{sub}\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}\right) e^{-\frac{q^2+m_\pi^2}{\Lambda^2}}$$

  - ▶ Separable contact interaction + local one-pion exchange interaction

- For simplicity: S-wave and $^1S_0$ NN interaction

- Solve the time-(in)dependent Schrodinger equation to get wave functions
- Time-independent method
  - ▶ Choose $\{\psi_{k_i}\}$ as inputs
- Time-dependent method
  - ▶ Initial wave functions

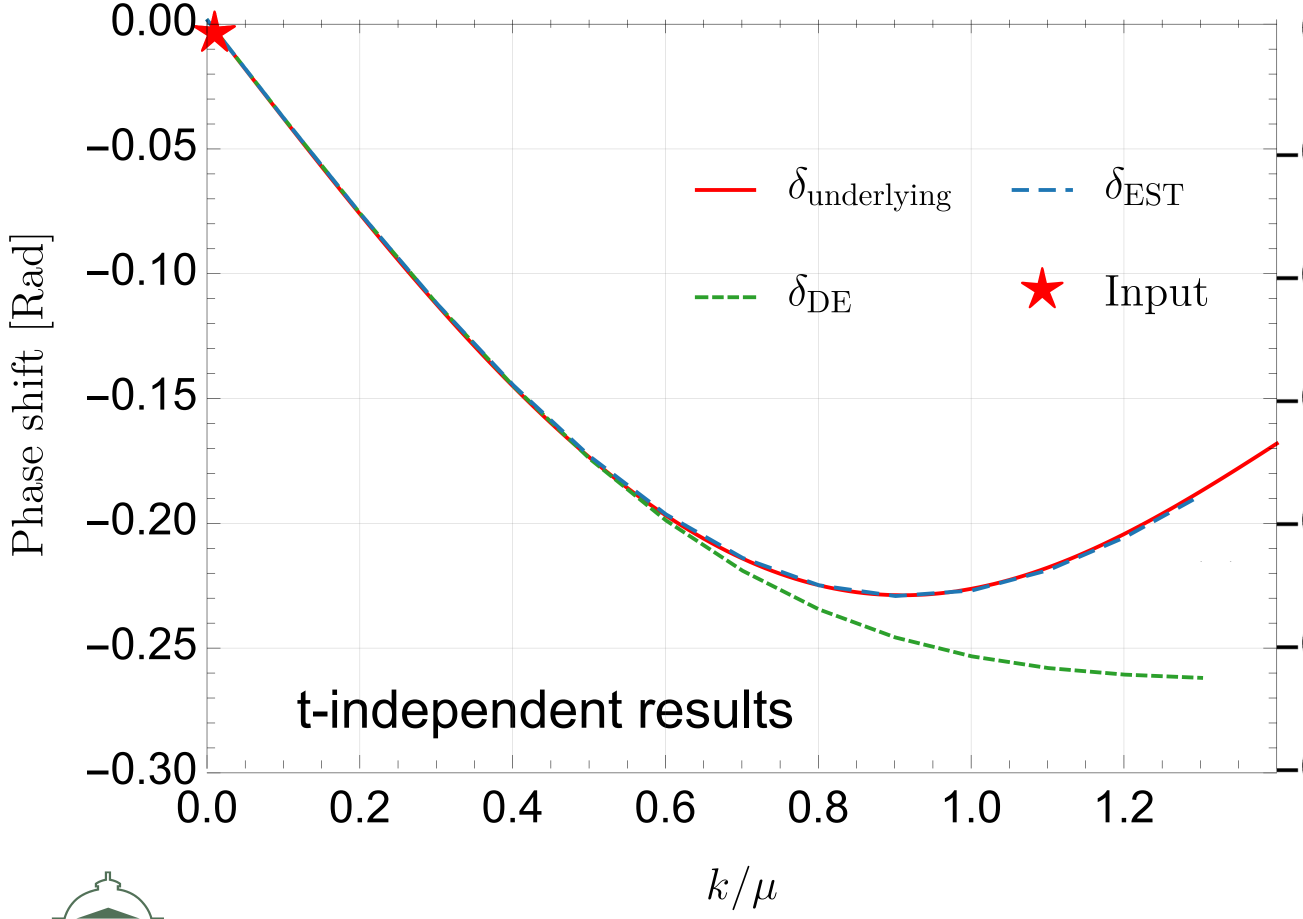
$$\tilde{R}(t=0,x) = \frac{\sigma^2 e^{-\sigma x}}{4\pi}$$

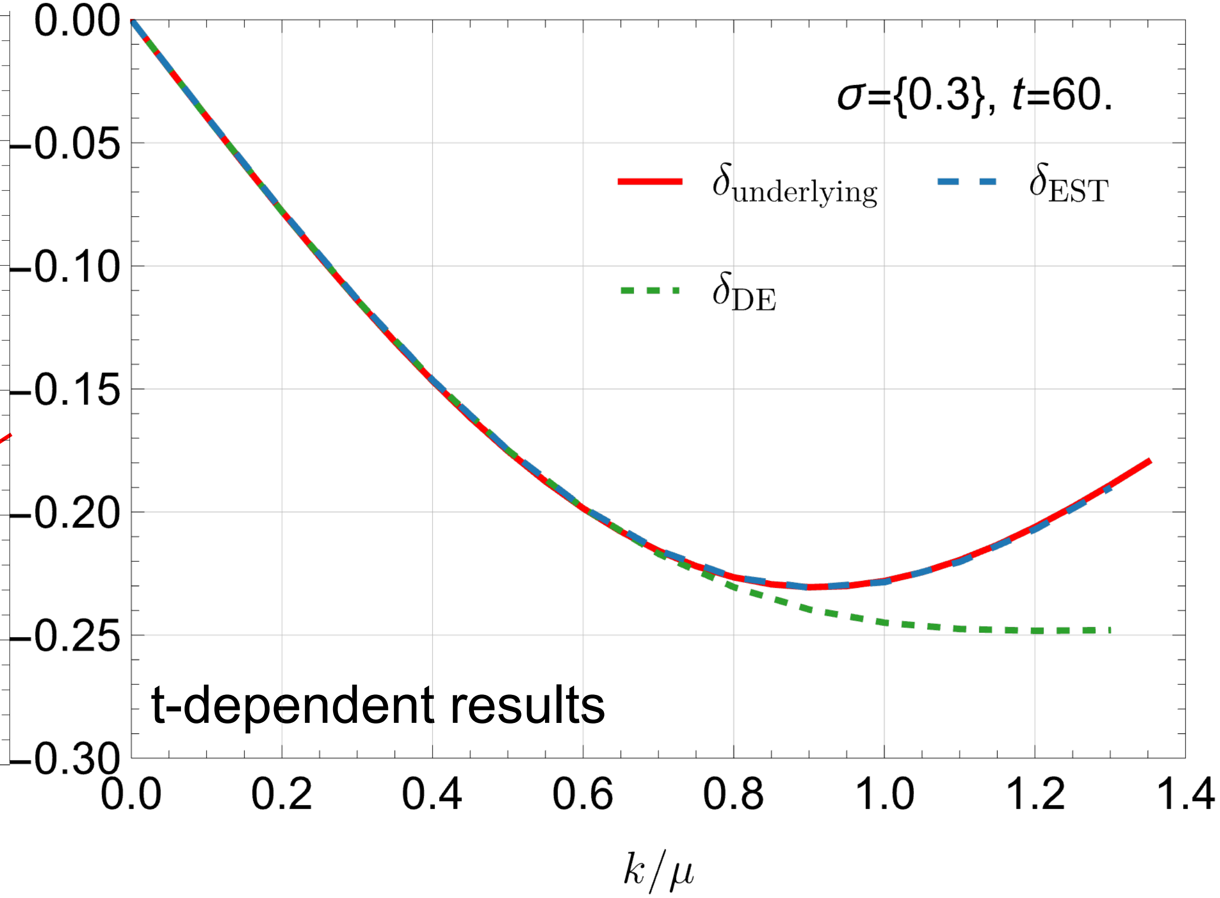
  - ▶ Evaluate t=60
  - ▶ Two $\sigma = \{0.3, 0.6\}$ as two inputs

Separatable interaction

- The EST methods give the accurate potential in LO
- The DE method is convergent

$$V(\boldsymbol{r},\boldsymbol{r}') = \omega \frac{e^{-\mu r}}{r}\frac{e^{-\mu r'}}{r'}$$





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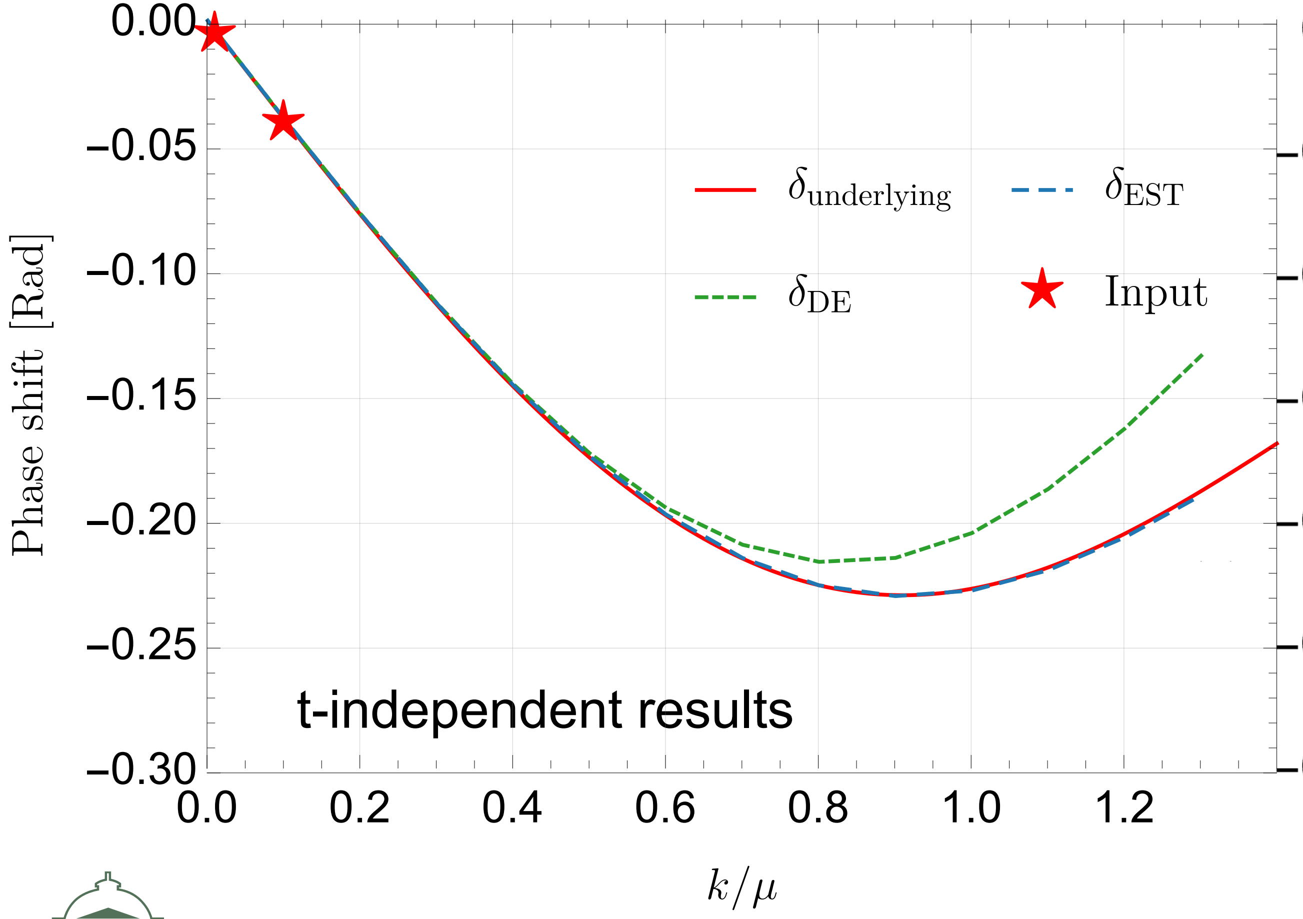
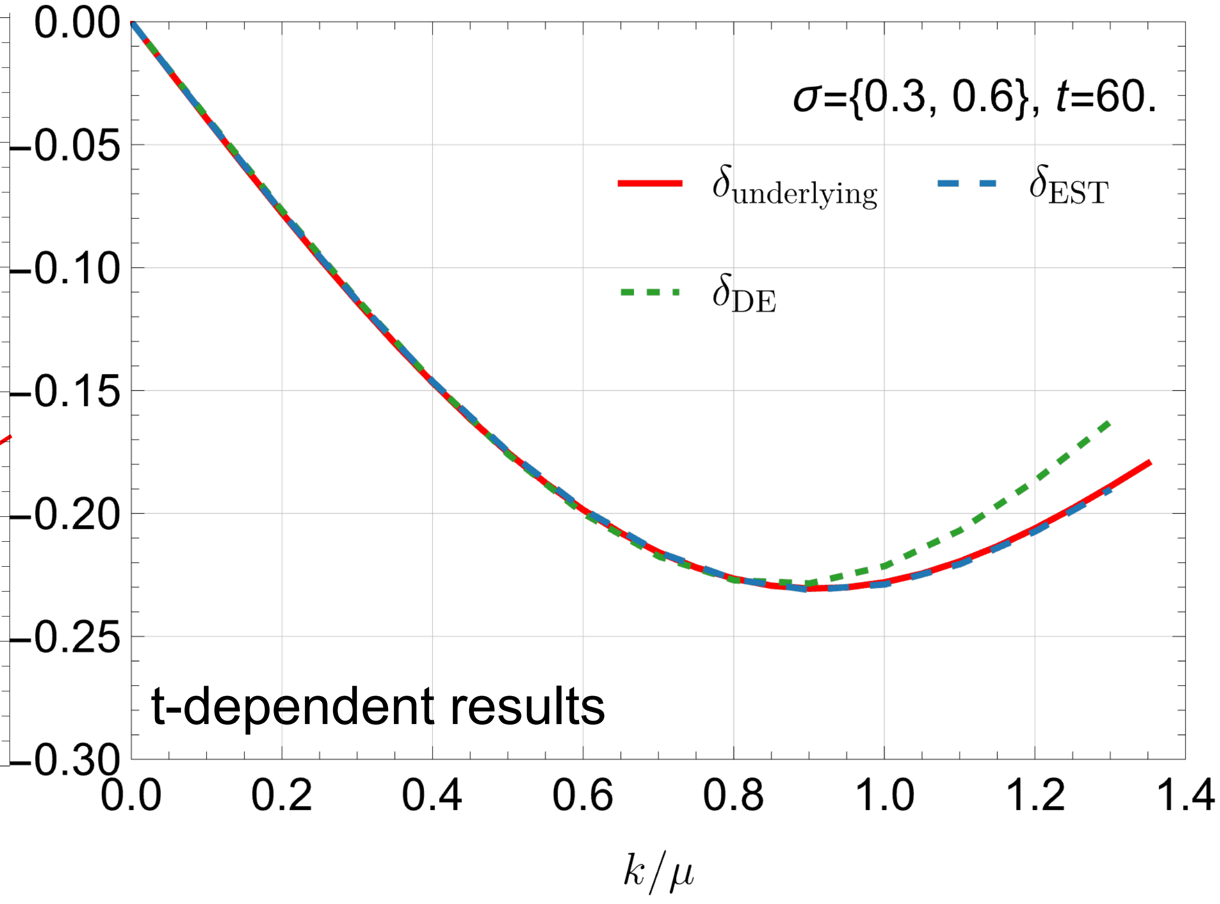


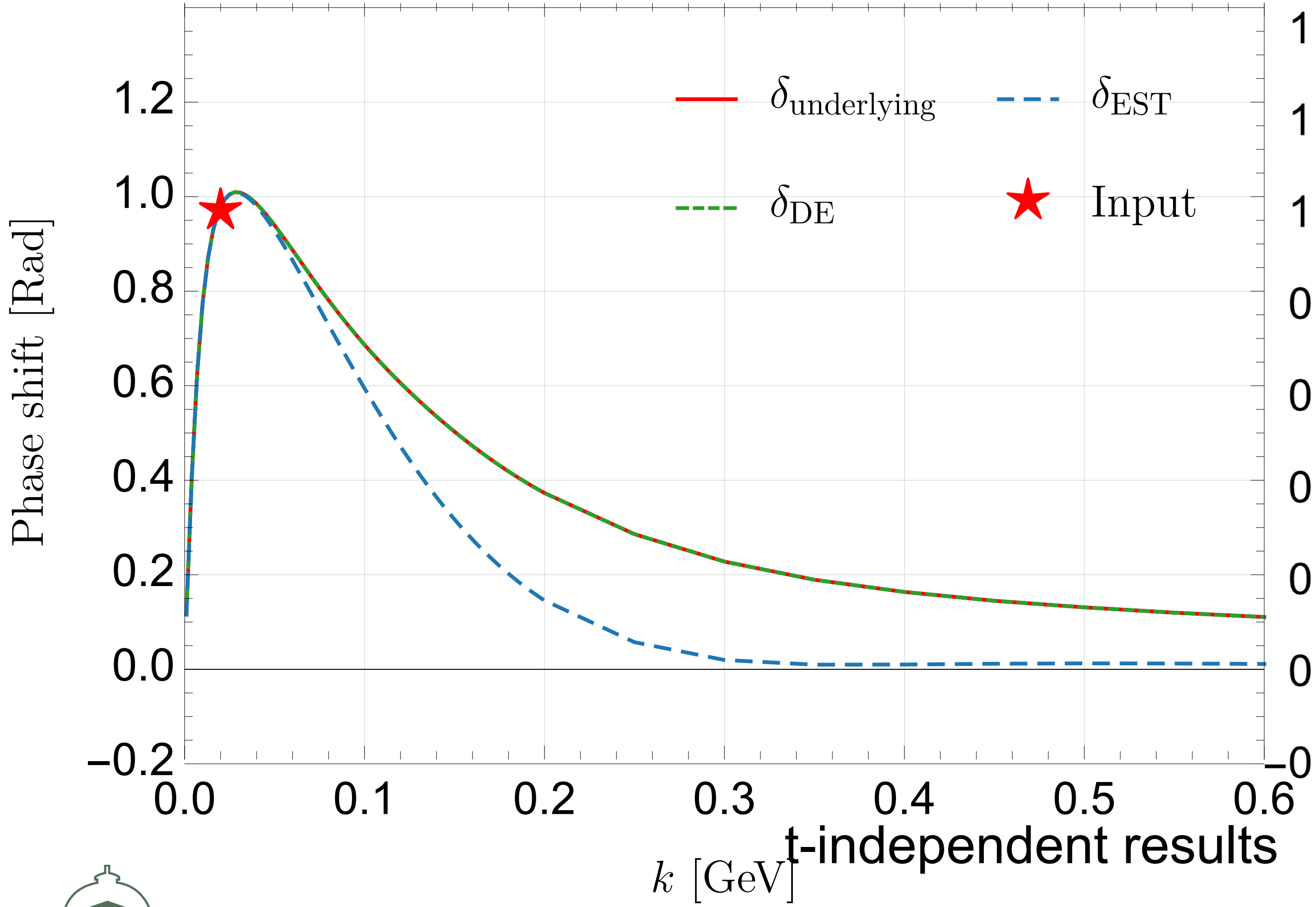


Local interaction

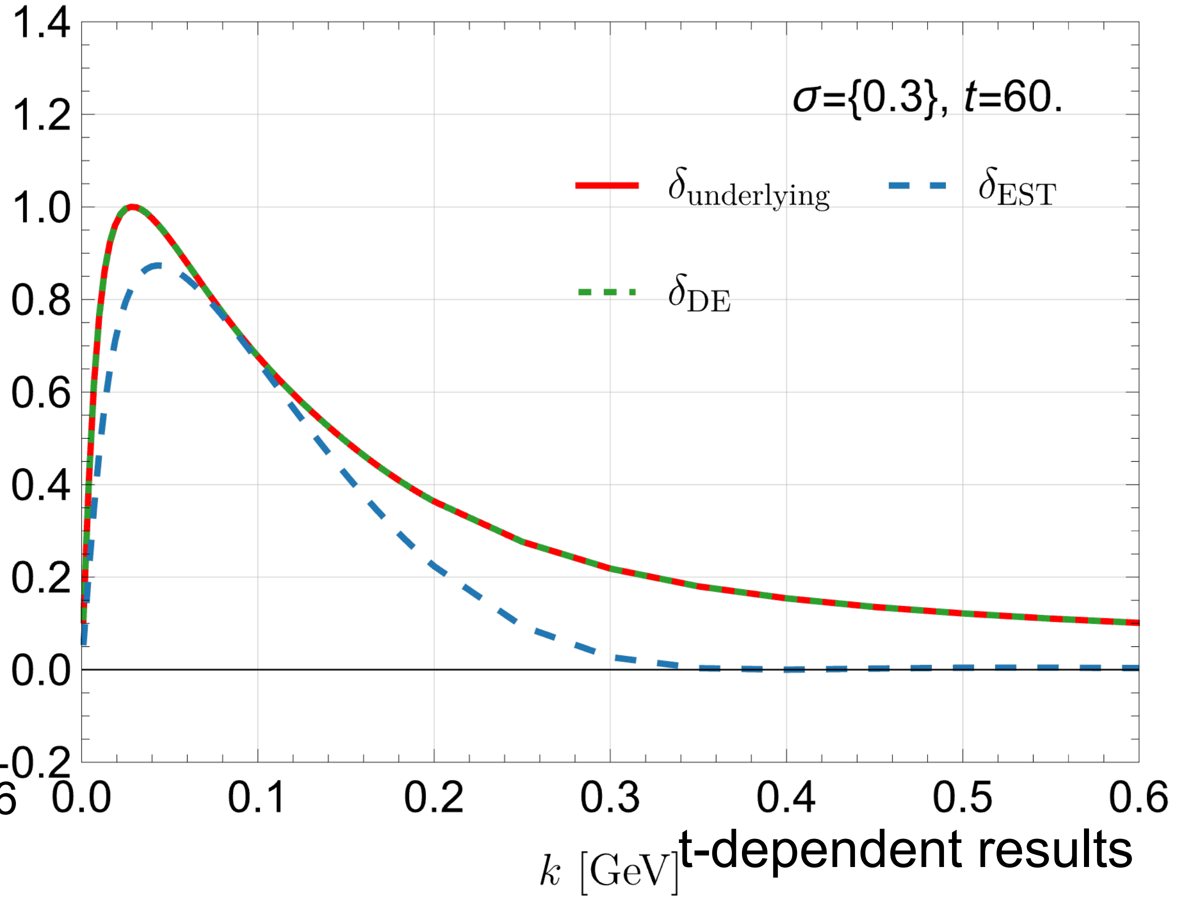
- The DE method gives the accurate results at LO
- Convergent EST results, not bad performance

~~$V_{ctc}(\boldsymbol{p},\boldsymbol{p}') = Ce^{-\frac{p^2+p'^2}{\Lambda^2}}$,~~

$$V_{ope}(\boldsymbol{q}) = -\frac{g_A}{4F_\pi^2}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{q}\boldsymbol{\sigma}_2\cdot\boldsymbol{q}}{\boldsymbol{q}^2+m_\pi^2} + C_{sub}\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}\right)e^{-\frac{q^2+m_\pi^2}{\Lambda^2}}$$



t-independent results



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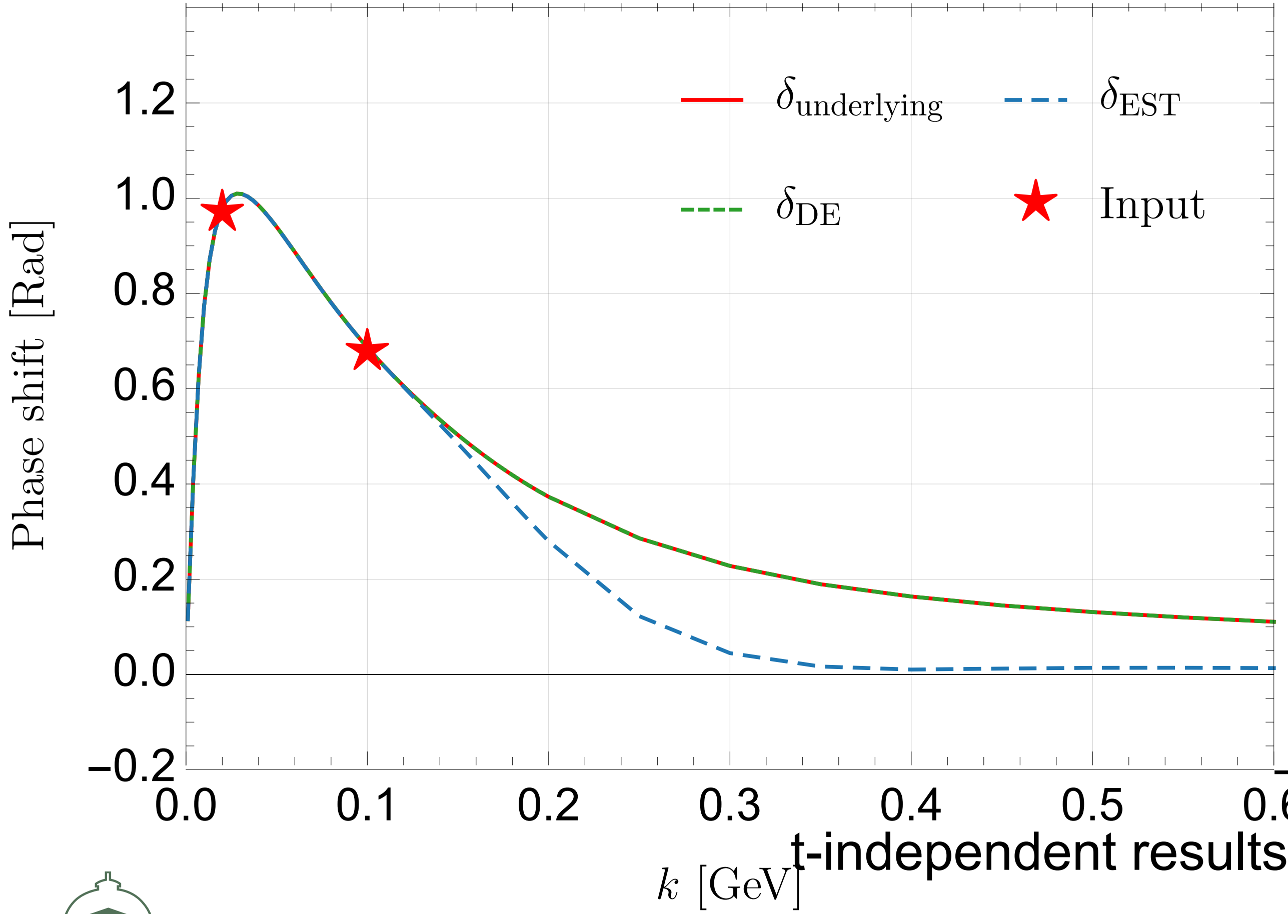
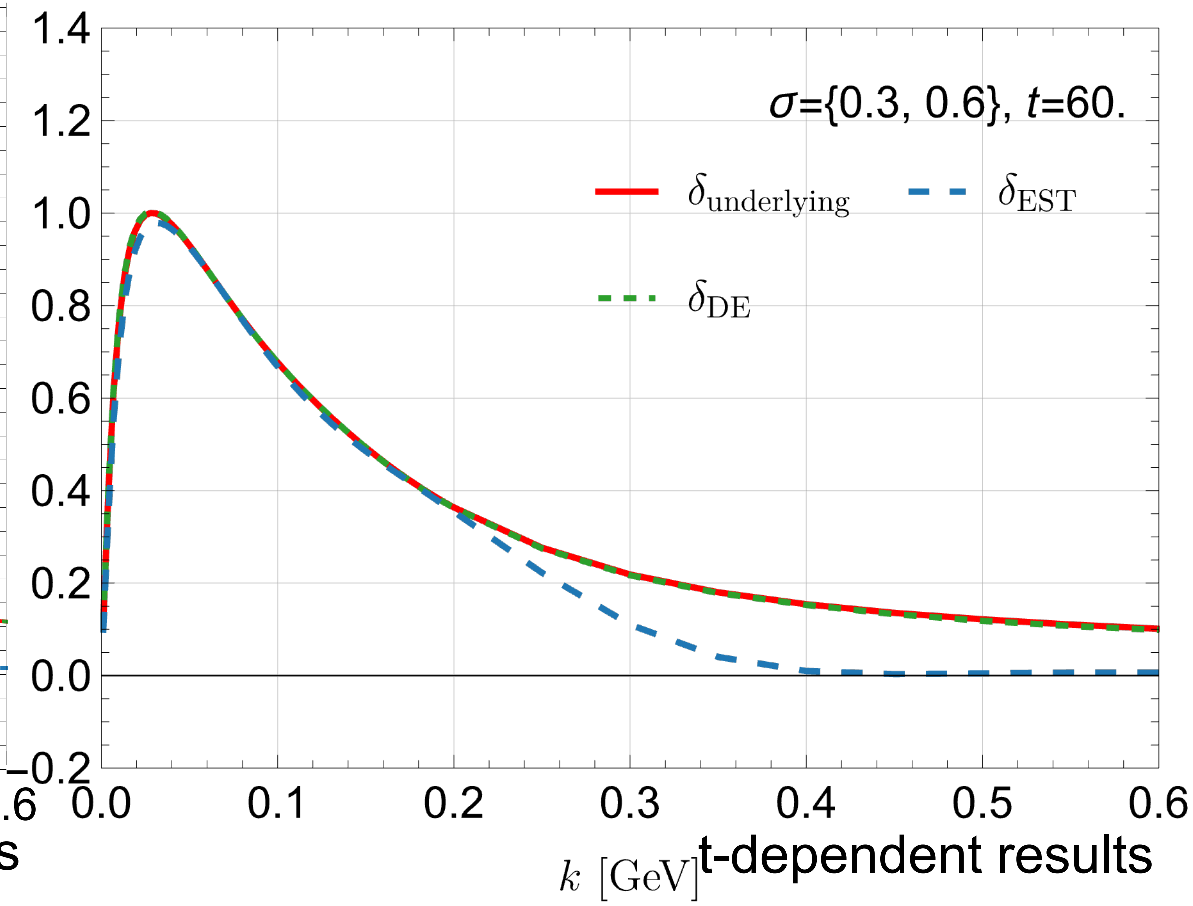


t-independent results

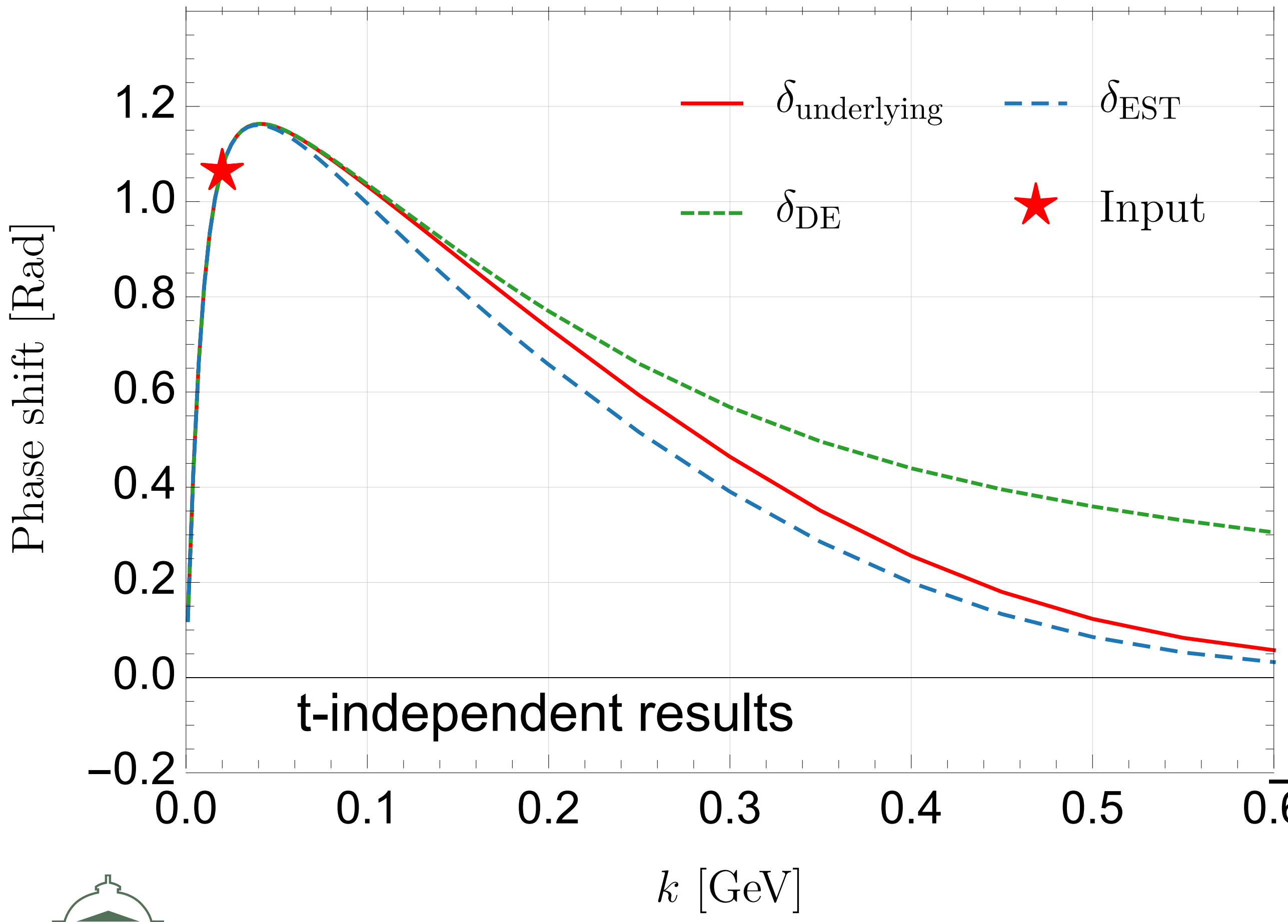


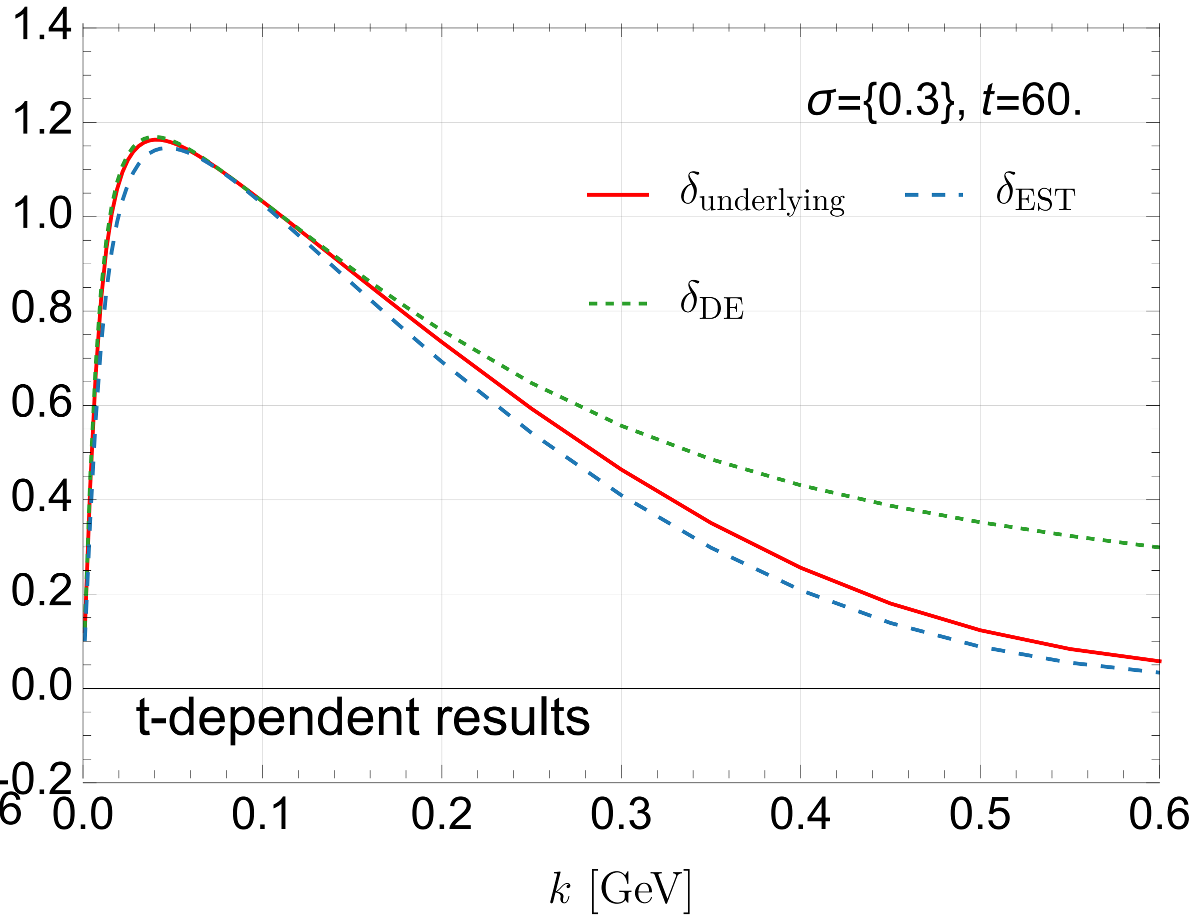
t-dependent results

- Including both separatable part and local part
- The performance of EST method is better
- In t-dependent methods, singular potential

$$V_{ctc}(\boldsymbol{p},\boldsymbol{p}') = Ce^{-\frac{p^2+p'^2}{\Lambda^2}},$$

$$V_{ope}(\boldsymbol{q}) = -\frac{g_A}{4F_\pi^2}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{q}\boldsymbol{\sigma}_2\cdot\boldsymbol{q}}{\boldsymbol{q}^2+m_\pi^2} + C_{sub}\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}\right)e^{-\frac{q^2+m_\pi^2}{\Lambda^2}}$$



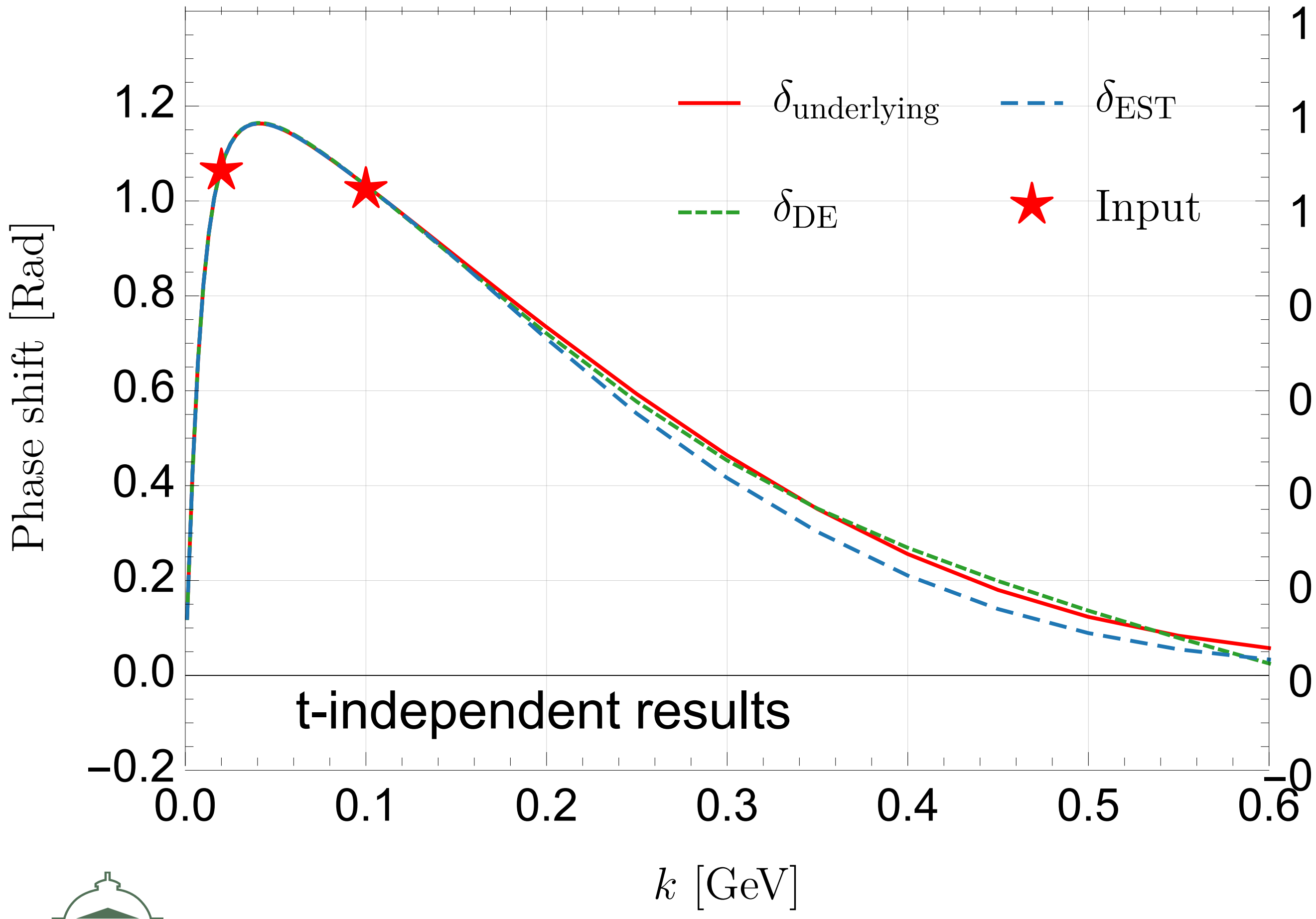


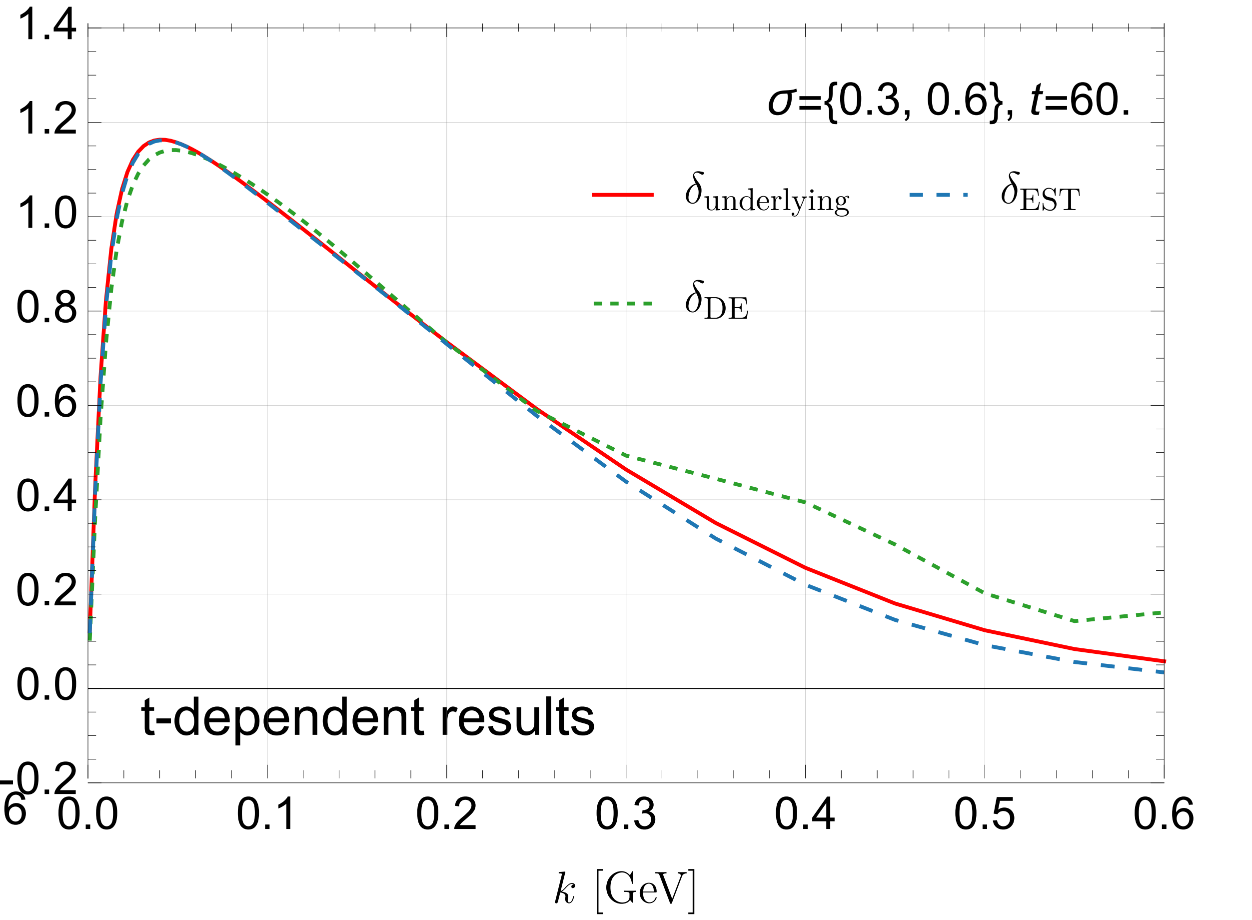
Physical interaction

- Including both separatable part and local part
- The performance of EST method is better
- In t-dependent methods, singular potential

$$V_{ctc}(\boldsymbol{p},\boldsymbol{p}') = Ce^{-\frac{p^2+p'^2}{\Lambda^2}},$$

$$V_{ope}(\boldsymbol{q}) = -\frac{g_A}{4F_\pi^2}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{q}\boldsymbol{\sigma}_2\cdot\boldsymbol{q}}{\boldsymbol{q}^2+m_\pi^2} + C_{sub}\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}\right)e^{-\frac{q^2+m_\pi^2}{\Lambda^2}}$$





Combine local and seperable parameterizaiton

- Chiral nuclear force
  - ▶ Long-range: pionic exchange interaction + Short-range: separable interaction

$$V_{ctc}(\boldsymbol{p},\boldsymbol{p}') = Ce^{-\frac{p^2+p'^2}{\Lambda^2}},$$

$$V_{ope}(\boldsymbol{q}) = -\frac{g_A}{4F_\pi^2}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{q}\boldsymbol{\sigma}_2\cdot\boldsymbol{q}}{\boldsymbol{q}^2+m_\pi^2} + C_{sub}\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}\right)e^{-\frac{q^2+m_\pi^2}{\Lambda^2}}$$

- Mixed strategy: $\Psi = \Psi_{\text{long}} + \Psi_{\text{resi}}$
  - ▶ Long-range part: local potential
  - ▶ Short-range part: EST method

- To be done: Check 3-body observable and peripheral phase shift

Contents

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Lüscher's formula

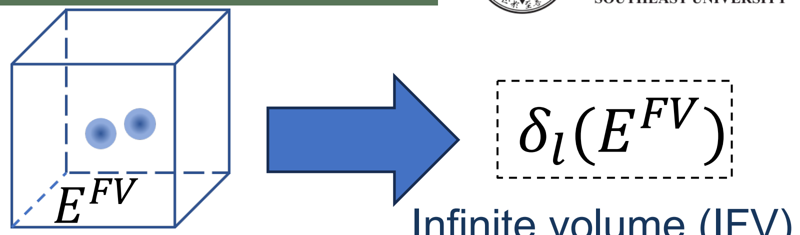
- Lüscher's formula:

Luscher:1990ux

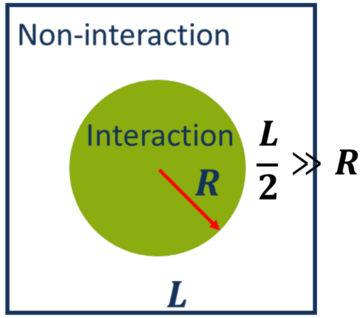
AKA : Lüscher Quantization conditions (LQCs)

$$\det\left[G_F^{-1}(L,E^{FV}) - K(E^{FV})\right] = 0$$

Kinematical term   K-matrix in IFV



- Quantization condition



Asymptotic behavior: for $r > R$

$$(\nabla^2 + k^2)\psi_k(r) = 0,$$

$$\psi_k(r) \sim \frac{e^{i\delta(k)}\sin[kr + \delta(k)]}{kr}.$$

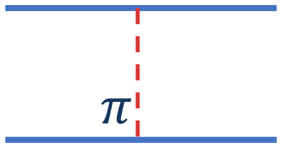
Periodic Boundary condition:

$$\psi(r) = \psi(r + L)$$

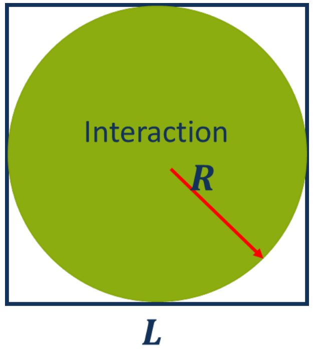
$$\boldsymbol{p} = \frac{2\pi}{L}\boldsymbol{n}, \qquad \boldsymbol{n} \in Z^3$$

- Limitations:
  - ▶ Exponentially suppressed effect: $e^{-L/R} \sim e^{-m_\pi L}$

    Require: $m_\pi L > 4 \Rightarrow L > 5.7$ fm
  - ▶ Left-hand cut (lhc) problem
  - ▶ Partial-wave mixing effects



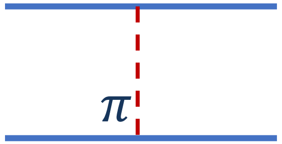
NN , D^*D systems…



Long-range interaction and small box???

Left-hand cut

- Left-hand cut (lhc) from the one-pion exchange interaction



Left-hand cut
on-shell pion

$$V(r) = \frac{e^{-mr}}{r}, \quad V(\vec{p}, \vec{p}') = \frac{1}{(\vec{p}' - \vec{p})^2 + m^2}$$

- Partial wave decomposition, e,g. S-wave

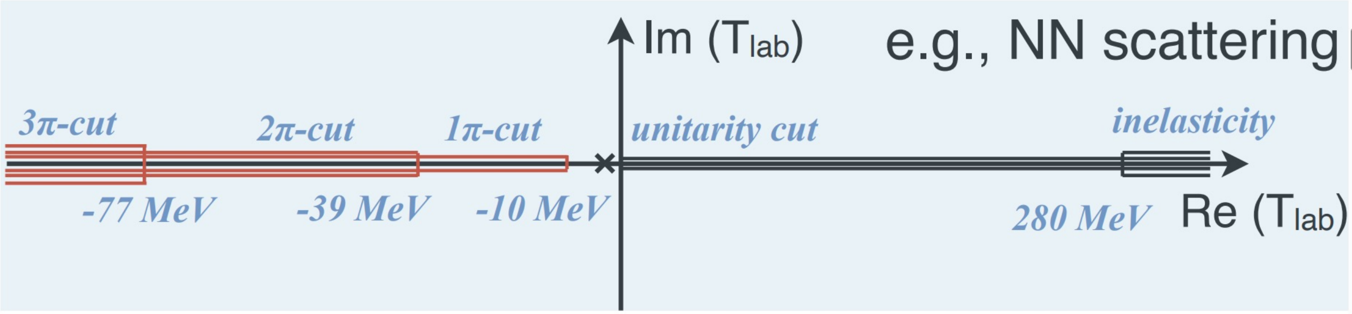
$$V_{l=0}(p,p') = \int_{-1}^{1} dz \frac{1}{p^2 + p'^2 - 2pp'z + m^2} = -\frac{1}{2pp'} \log\left(\frac{(p-p')^2 + m^2}{(p+p')^2 + m^2}\right)$$

- On-shell $p = p' = k, \;\; k^2 = 2\mu E$

$$V_{l=0}(k,k) = \int_{-1}^{1} dz \frac{1}{2k^2(1-z) + m^2}, \qquad 2k^2(1-z) + m^2 = 0 \Rightarrow z = \frac{m^2}{2k^2} + 1, \qquad -1 < z < 1 \Rightarrow k^2 < -\frac{m^2}{4}$$

on-shell pion

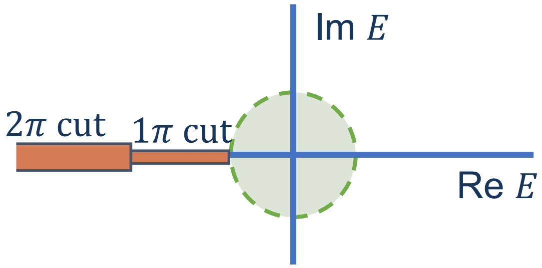
▶ Branch point: $k^2 < -\frac{m^2}{4}$



Left-hand cut problem

- lhc in the IFV
  - ▶ Effective range expansion (ERE):

$$K^{-1}(p) = p\cot\delta(p) = \frac{1}{a} + \frac{1}{2} r p^2 + \cdots$$



  - ▶ Radius of convergence of ERE

    Meng-Lin's talk

    NN: Baru:2015ira, Baru:2016evv

    DD^*: Du:2023hlu

- lhc problem of Lüscher formula

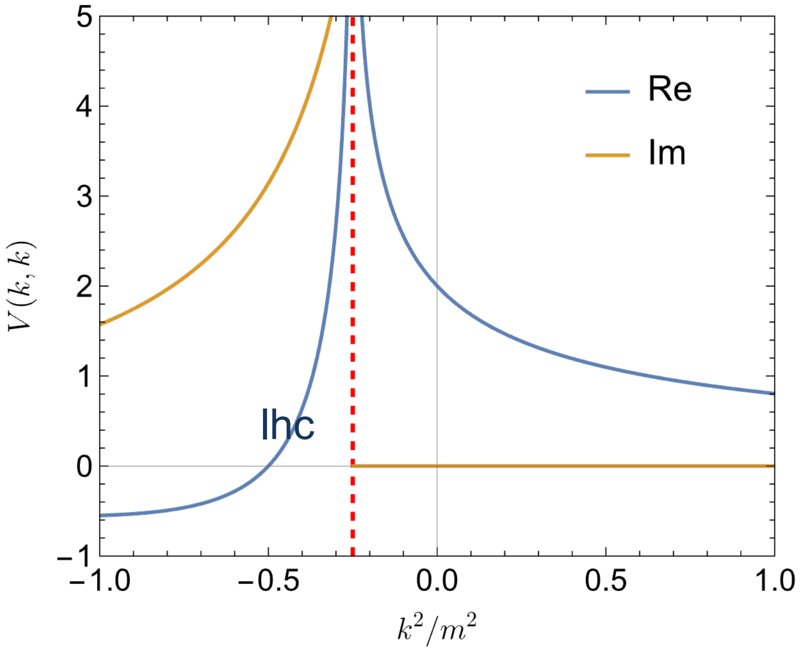
$$\det\left[G_F^{-1}(L,E) - K(E)\right] \neq 0$$

Real — K-matrix in the IFV

$$K = V + V G^{\mathcal{P}} K$$

  - ▶ For $k^2 > -\frac{m^2}{4}$, K-matrix is real
  - ▶ For $k^2 < -\frac{m^2}{4}$, $\text{Im}\, K \neq 0$

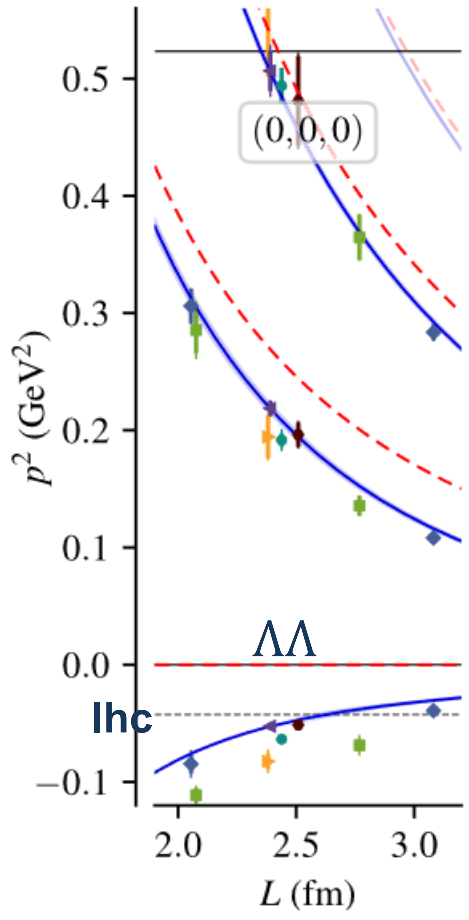
    Jin-Yi's talk



Left-hand cut problem in LQCD

Green:2021qol

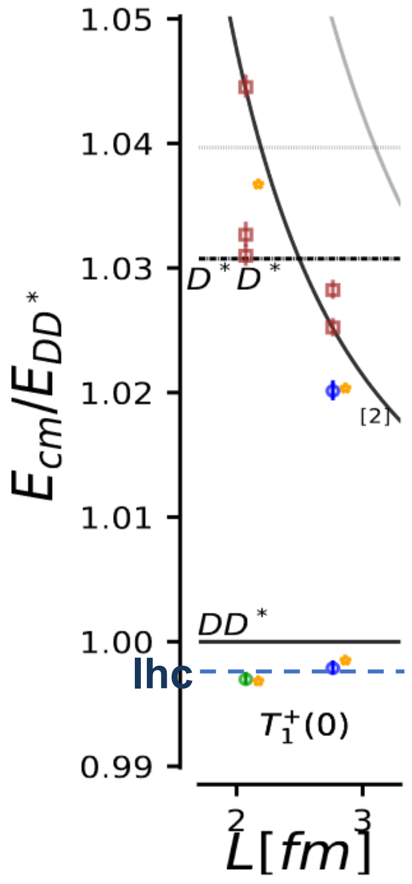
$m_\pi = m_K \approx 420$ MeV



H-dibaryons ($udsuds$)

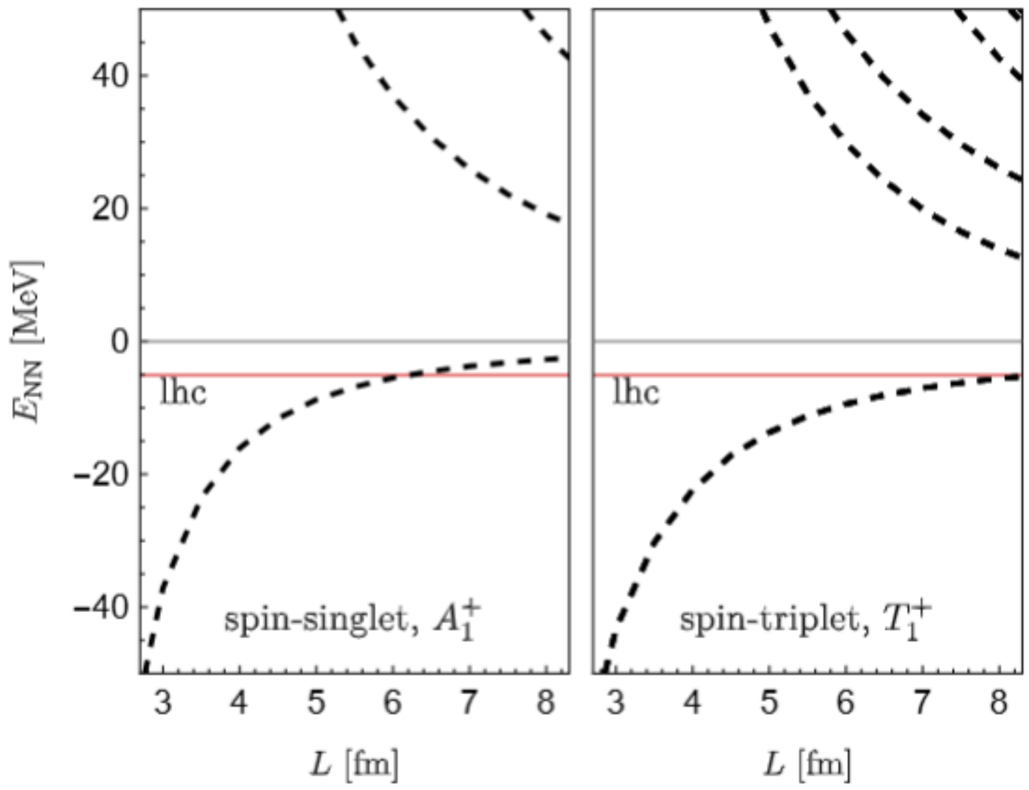
Padmanath:2022cvl

$m_\pi = 280$ MeV



DD^* (T_{cc})

$m_\pi = 137$ MeV



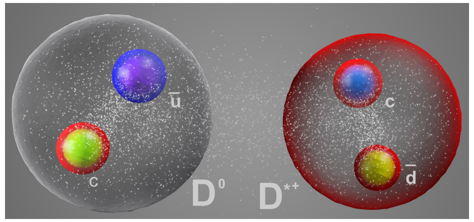
NN systems

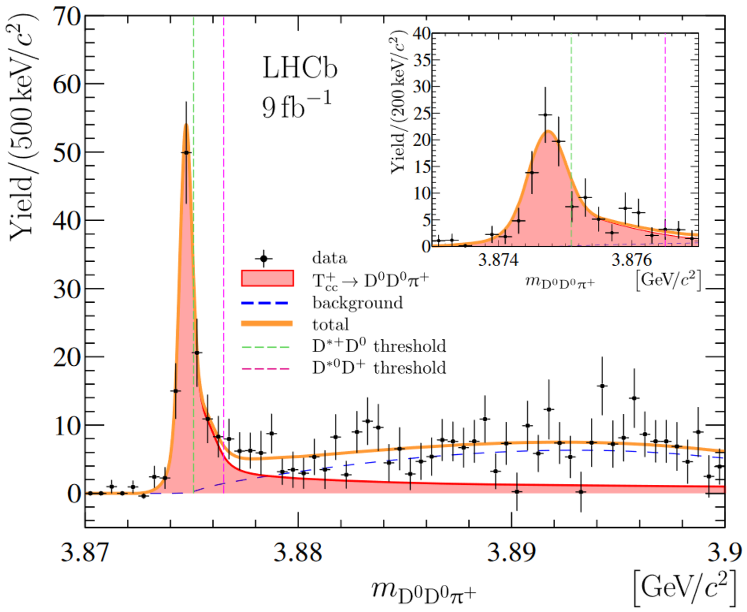
$L > 8$ fm to get rid of the lhc problem

$T_{cc}(3875)$

- $T_{cc}(3875)^+$ was observed in 3-body final states: $D^0 D^0 \pi^+$
  - ▶ $cc\bar{u}\bar{d}$, the 1st open double charm tetraquark state
  - ▶ close to $D^0 D^{*+}$ threshold, good candidates of molecular state
  - ▶ Isospin violation effect
  - ▶ 3-body effect

  Du:2021zzh, Meng:2021jnw…





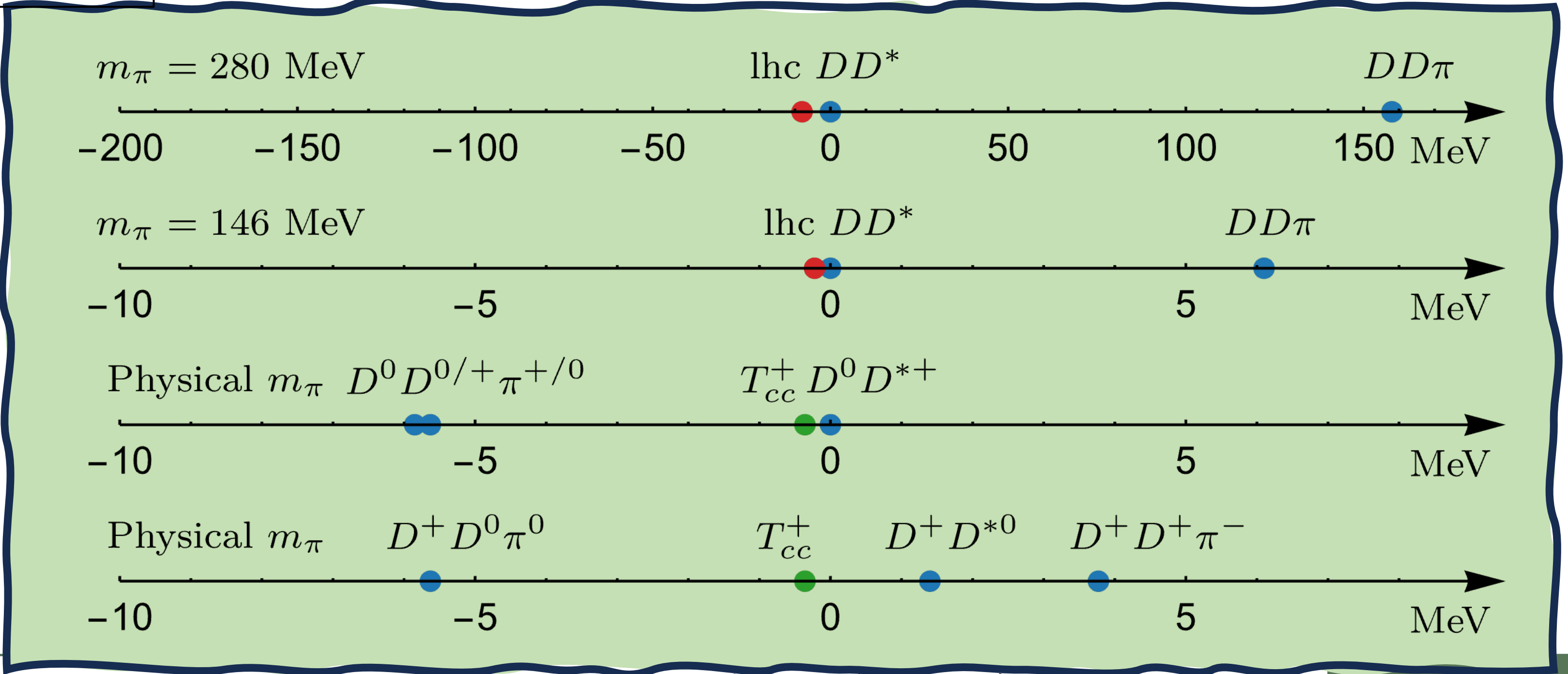
LHCb:2021vvq, LHCb:2021auc

- Left-hand cut
- Three-body effect

Finite volume

Infinite volume

- m_q-dependence Abolnikov:2024key
  - ▶ Pion mass
  - ▶ Cuts
  - ▶ Pole
- m_Q-depedence Collins:2024sfi
  - ▶ From T_{cc} to T_{bb}
  - ▶ From molecule to compact



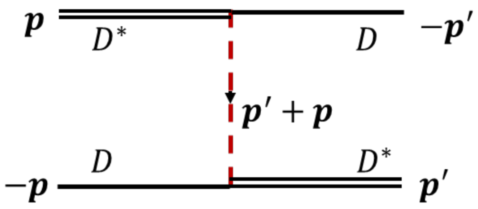
T_{cc} is a playground

T_{cc} lattice QCD simulations

- LQCD setting: $m_\pi \approx 280$ MeV, $m_D \approx 1927$ MeV, $m_{D^*} \approx 2049$ MeV, $L \approx 2.07, 2.76$ fm, $a \approx 0.086$fm

Padmanath:2022cvl

- Some quick estimations
  - $m_{\rm eff}^2 = m_\pi^2 - (m_{D^*} - m_D)^2 > 0$, $m_{\rm eff} \approx 252$ MeV
  - $p_{\rm lhc}^2 \approx -\left(\frac{m_{\rm eff}}{2}\right)^2 = -(126\text{ MeV})^2$
  - $p_{\rm rhc3}^2 \approx 2\mu_{DD^*}(2m_D + m_\pi - m_D - m_{D^*}) \approx (560\text{ MeV})^2$

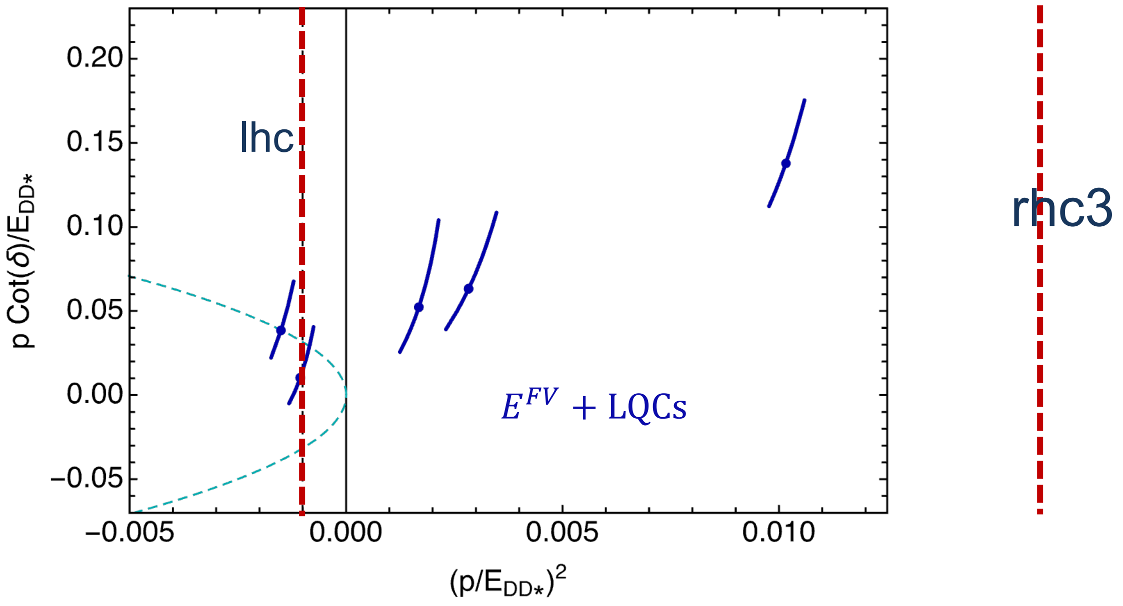


$$\frac{-1}{k^2 + m_\pi^2 - k_0^2 - i\epsilon} = \frac{-1}{k^2 + m_{\rm eff}^2 - i\epsilon}$$

- A conventional procedure to the IFV:
  - Using Lüscher formula to get phase shift
  - Use ERE to parameterize K-matrix
- Conclusion: virtual states
- Limitations
  - $m_{\rm eff}L = 2.6, 3.5$

  Exponential suppressed effect can be important
  - left-hand cut

  Lüscher formula, effective range expansion



Other lattice results: Cheung:2017tnt, Junnarkar:2018twb, Chen:2022vpo, Lyu:2023xro, Whyte:2024ihh…

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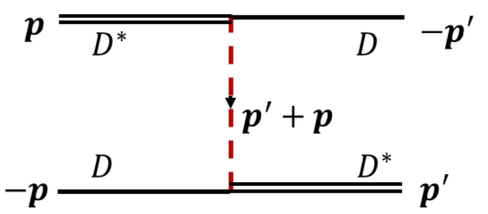

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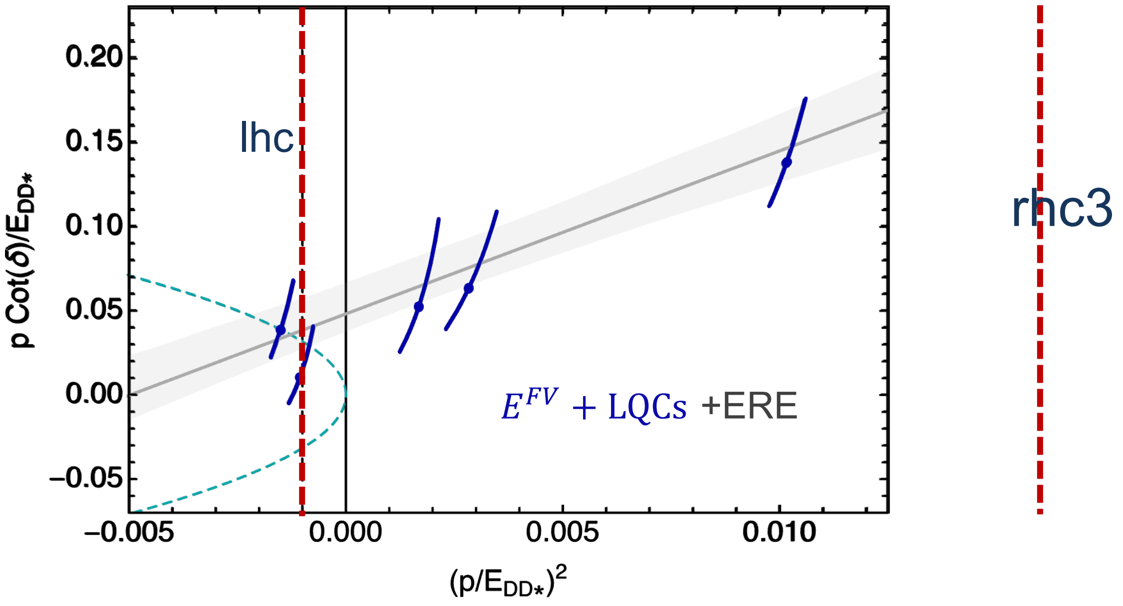
~~Lüscher formula~~, ~~effective range expansion~~



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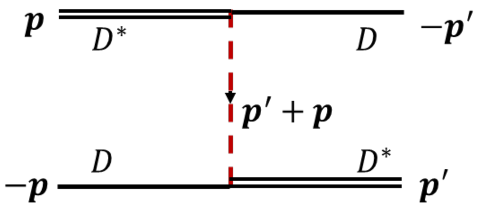

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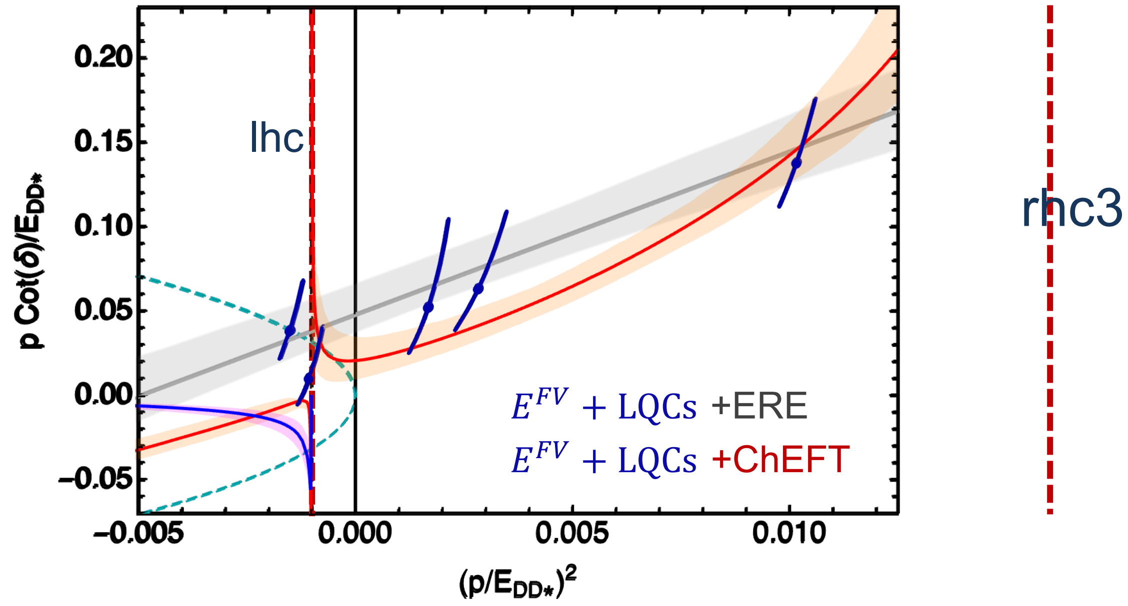

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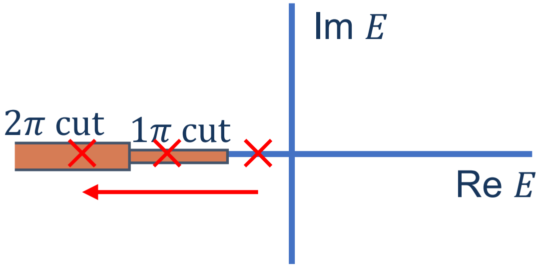
Contents

Key point: off-shell effect

- Lüscher formula:
  - ▶ The E_{FV}s are only related to the **on-shell** T-matrix
  - ▶ **The off-shell** effect is exp. suppr. and thus neglectable
- The lhc problem of the Lüscher formula: off-shell effect, exp. suppr. effect
- Schrödinger Eq. in the IFV to get the **bound state** solutions

$$\frac{\boldsymbol{p}^2}{2\mu}\psi(\boldsymbol{p}) + \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} V(\boldsymbol{p},\boldsymbol{p}')\psi(\boldsymbol{p}) = E\psi(\boldsymbol{p})$$

Off-shell, for $E < 0$

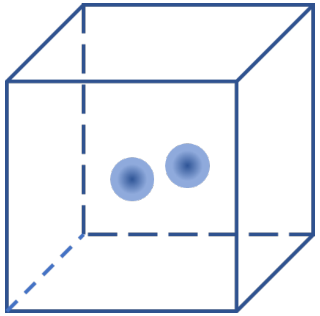


  - ▶ Works well even below the left-hand cut
  - ▶ For the $p, p' > 0$, no lhc in potential

$$V_{l=0}(p,p') = \int_{-1}^{1} dz \frac{1}{p^2 + p'^2 - 2pp'z + m^2} = -\frac{1}{2pp'} \log\left(\frac{(p-p')^2 + m^2}{(p+p')^2 + m^2}\right)$$

- FV energy levels are "bound states" trapped by the potential well

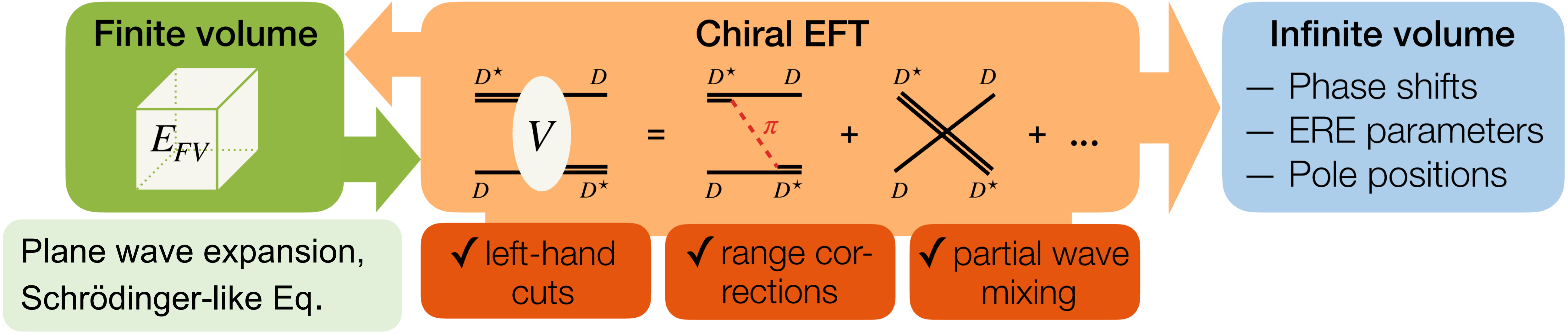
$$\int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \to \frac{1}{L^3}\sum_{\boldsymbol{p}_n}$$



- Basic idea: using V to connected FV and IFV; FV effect: Schrödinger-like Eq.

Our strategy

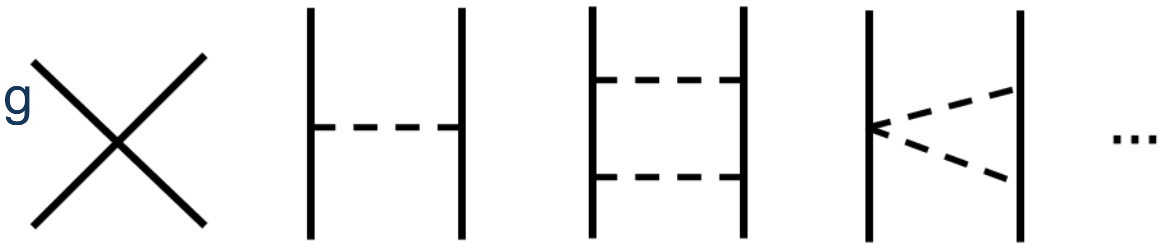
- Hamiltonian method in plane wave basis + Chiral effective field theory



J.-J. Wu, T.-S. H. Lee, et al, Phys. Rev. C **90**, 055206 (2014).
L. Meng and E. Epelbaum, JHEP **10**, 051 (2021) (plane wave basis + irrps.)

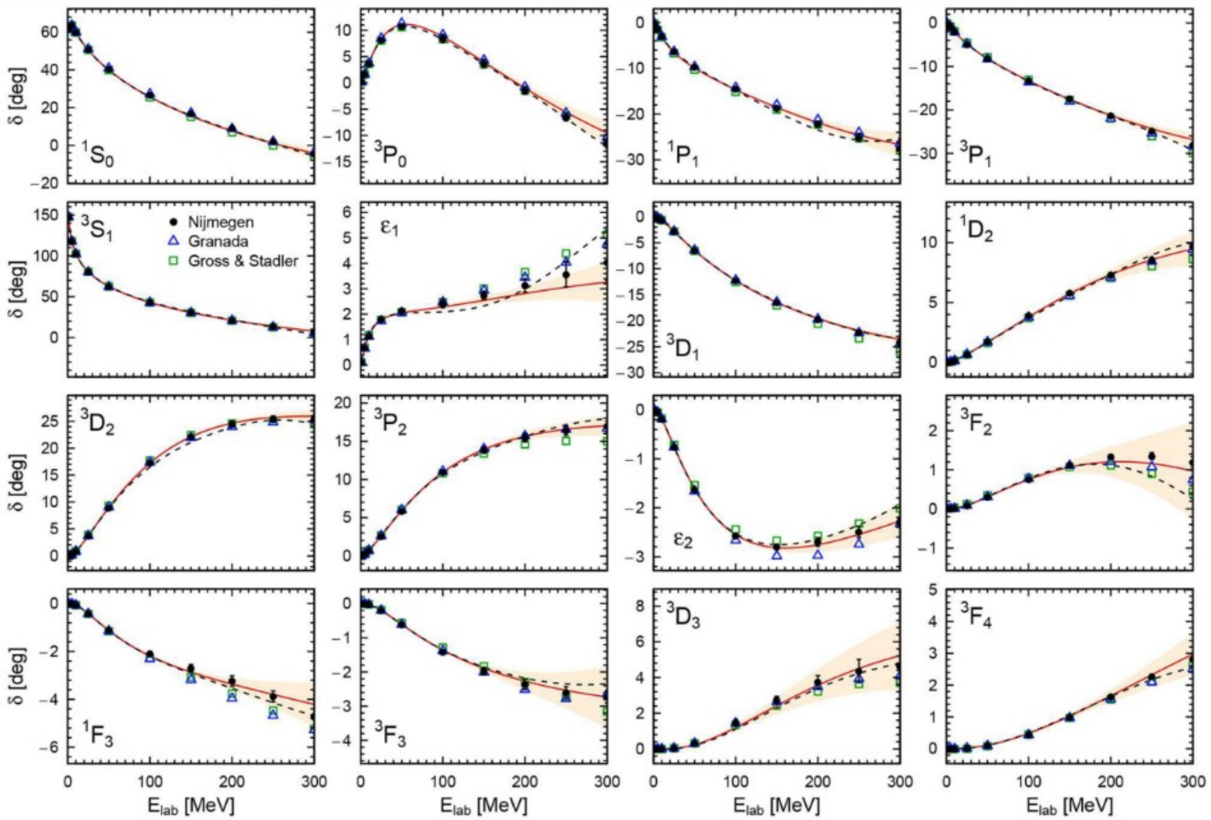
Chiral EFT

- Symmetry from QCD
  - ▶ Chiral symmetry and its spontaneous breaking
- Weinberg power counting
  - ▶ Systemic, controllable truncation error



Reinert:2017usi

$$V(\vec{p}', \vec{p}) = V_{\text{contact}} + V_{1\pi} + V_{2\pi}$$



- Great success in the nuclear force
- Semilocal momentum-space regularization

$$V_{1\pi}(\vec{p}', \vec{p}) = -\frac{g_A^2}{4F_\pi^2}\left(\frac{\vec{\sigma}_1\cdot\vec{q}\vec{\sigma}_2\cdot\vec{q}}{q^2+m_\pi^2} + C(m_\pi)\vec{\sigma}_1\cdot\vec{\sigma}_2\right)e^{-\frac{q^2+m_\pi^2}{\Lambda^2}}$$

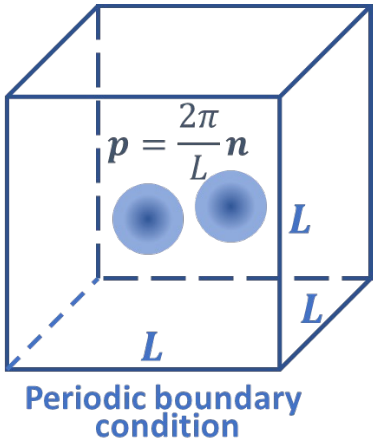
- Long-range interaction: $V_{1\pi}$ is known
- Short-range interaction: contact interaction
  - ▶ Unknown low energy constants (LECs)
  - ▶ fitting lattice QCD data

Hamiltonian method in plane wave basis

- Boundary conditions in the cubic box

$$\psi(\mathbf{x}_1, \mathbf{x}_2) = \psi(\mathbf{x}_1 + \mathbf{n}_1 L, \mathbf{x}_2 + \mathbf{n}_2 L)$$

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{P}, \quad \boldsymbol{p}_1 = \frac{2\pi}{L}\boldsymbol{n}, \quad \boldsymbol{P} = \frac{2\pi}{L}\boldsymbol{d}, \qquad \boldsymbol{n}, \boldsymbol{d} \in Z^3$$



- The rotation symmetry is broken: $SO(3) \to O_h$
  - ► $\{l, m\}$ are not good quantum numbers to label states
  - ► Partial wave mixing, for $l \neq l'$ and $m \neq m'$,

$$\langle lm|H^{FV}|l'm'\rangle \neq 0$$

  - ► The FV energy from lattice: irreducible representations (irreps.) of O_h

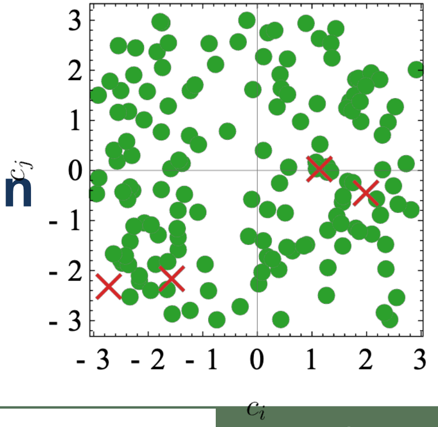
$$\{l, m\} \to \{A_1, A_2, E, T_1, T_2\}$$

- Why not use the **plane wave (with discrete momentum)** basis directly?
- Finite volume energy levels

$$\det\left(\mathbb{G}^{-1} - \mathbb{V}\right) = 0 \to \det\left(\mathbb{H} - E\mathbb{I}\right) = 0,$$

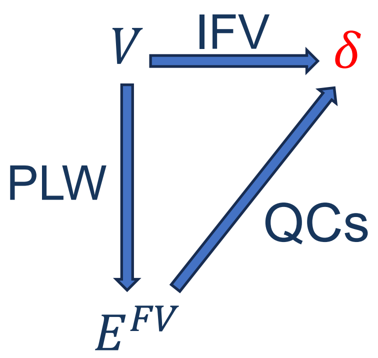
$$\mathbb{H} \Rightarrow \text{diag}\{\mathbb{H}_{\Gamma_i}, \mathbb{H}_{\Gamma_j}, \ldots\} \Rightarrow \quad \mathbb{H}_\Gamma \boldsymbol{v} = E_\Gamma \boldsymbol{v} \qquad \Gamma\text{: irreps.}$$

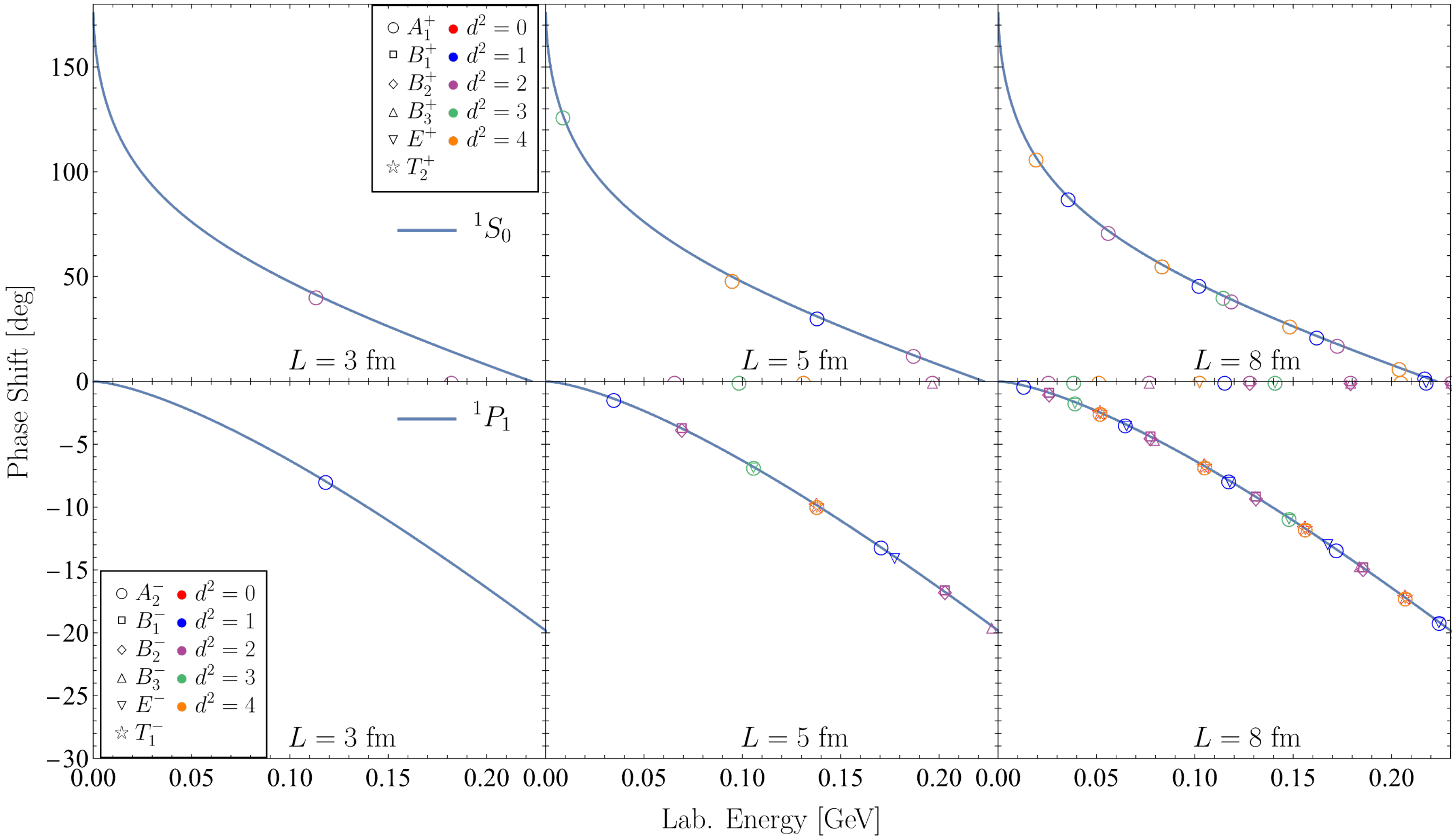
- Accelerate calculation: subspace learning, specifically **eigenvector continuation**
- Extra advantage: **partial wave mixing effect**



Benchmark: contact interaction

- Contact interaction: $V(\boldsymbol{p},\boldsymbol{p}') = C_S + C_1\boldsymbol{q}^2 + C_2\boldsymbol{k}^2$
- Only contribute to S-wave and P-wave





Benchmark: chiral EFT, partial wave mixing

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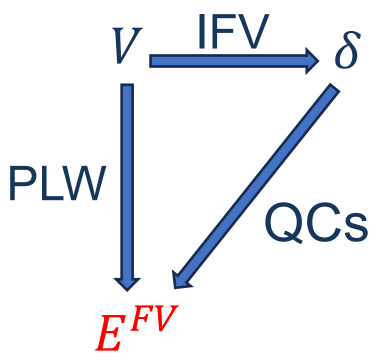
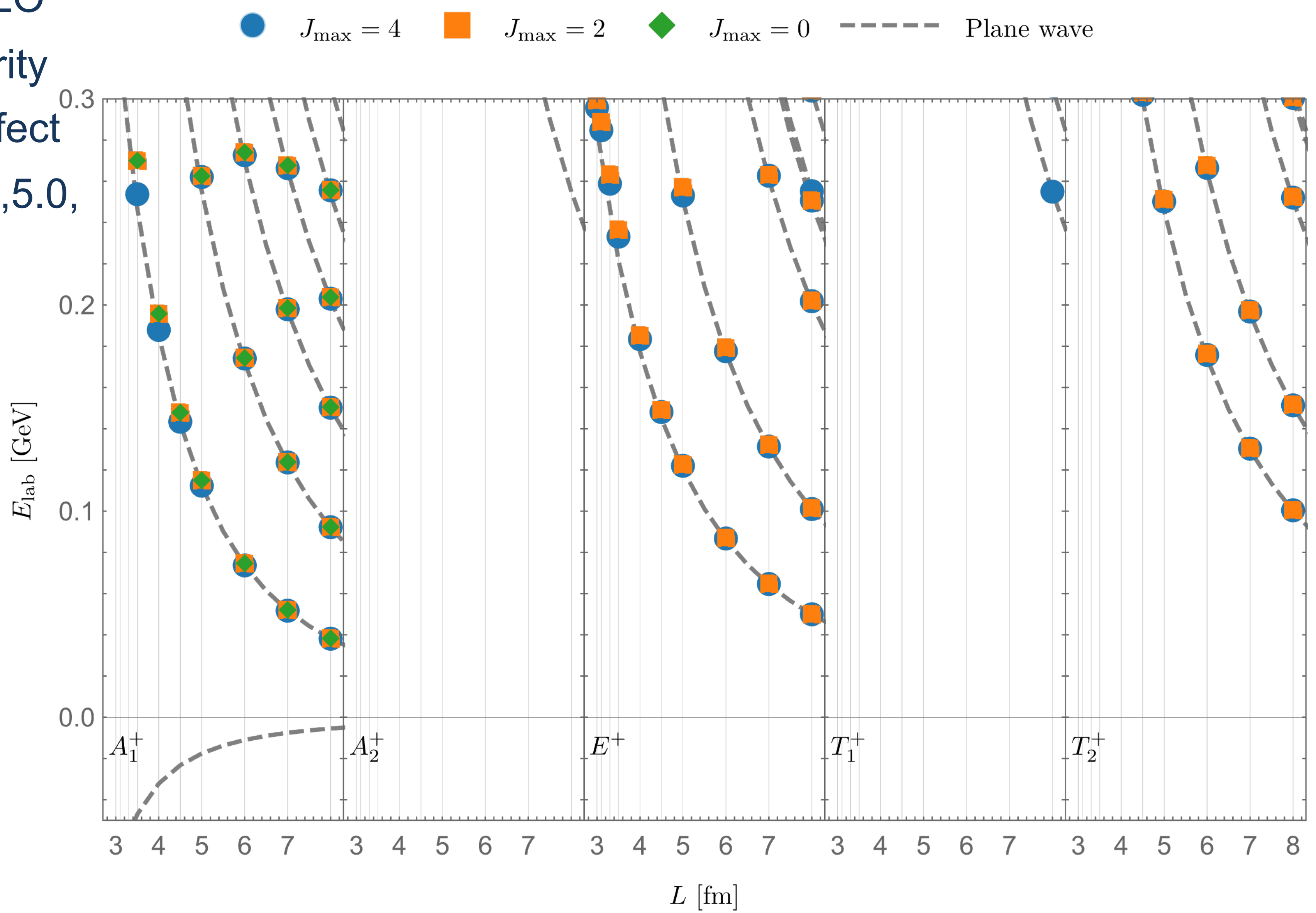
- ChEFT nuclear force: NNLO
- S=0, $d = (0,0,0)$, even parity
- QCs with partial mixing effect
- L={ 3.0,3.1,3.3,3.5,4.0,4.5,5.0, 6.0,7.0,8.0} fm



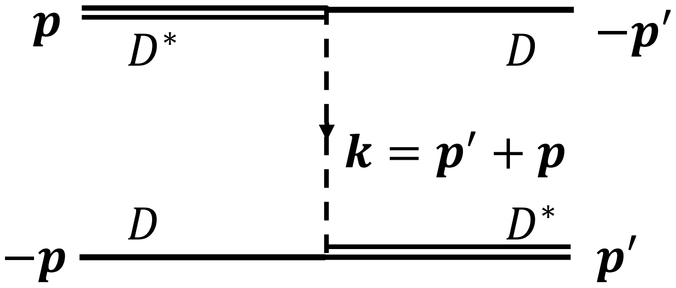


- The discrepancy
  - ▶ Small box
  - ▶ Small $J_{\max}$ truncation

- The energy levels of $A_1^-(0)$ is high, relativistic formalism

$$T(\boldsymbol{p},\boldsymbol{p}')=V(\boldsymbol{p},\boldsymbol{p}')+\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}V(\boldsymbol{p},\boldsymbol{q})\frac{1}{2w_1w_2}\frac{(w_1+w_2)}{P_0^2-(w_1+w_2)^2+i\epsilon}T(\boldsymbol{q},\boldsymbol{p}')$$

- Interaction:
  - ▶ Contact terms: LO and NLO $^3S_1$, NLO $^3P_0$
  - ▶ One-pion-exchange interaction



- Replace integral into summation to get

Li:2021mob

$$\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\to\mathcal{J}\int\frac{d^3\boldsymbol{q}_{box}}{(2\pi)^3}\to\mathcal{J}\sum_{\boldsymbol{n}}\frac{1}{L^3}\qquad\mathbb{T}=\mathbb{V}+\mathcal{J}\mathbb{V}.\mathbb{G}.\mathbb{T}$$

$\mathcal{J}$: is the Jacobian determinant of the Lorentz boost

- Get the poles

$$\det\left(\mathbb{H}-\lambda\mathbb{I}\right)=0\to\mathbb{H}\boldsymbol{v}=\lambda\boldsymbol{v},$$

$\Lambda = 0.9$ GeV, only contact terms

Padmanath:2022cvl

- LQCs+ ERR: 4 paras.

  $a_{^3S_1}, r_{^3S_1}, a_{^3P_0}, r_{^3P_0}$

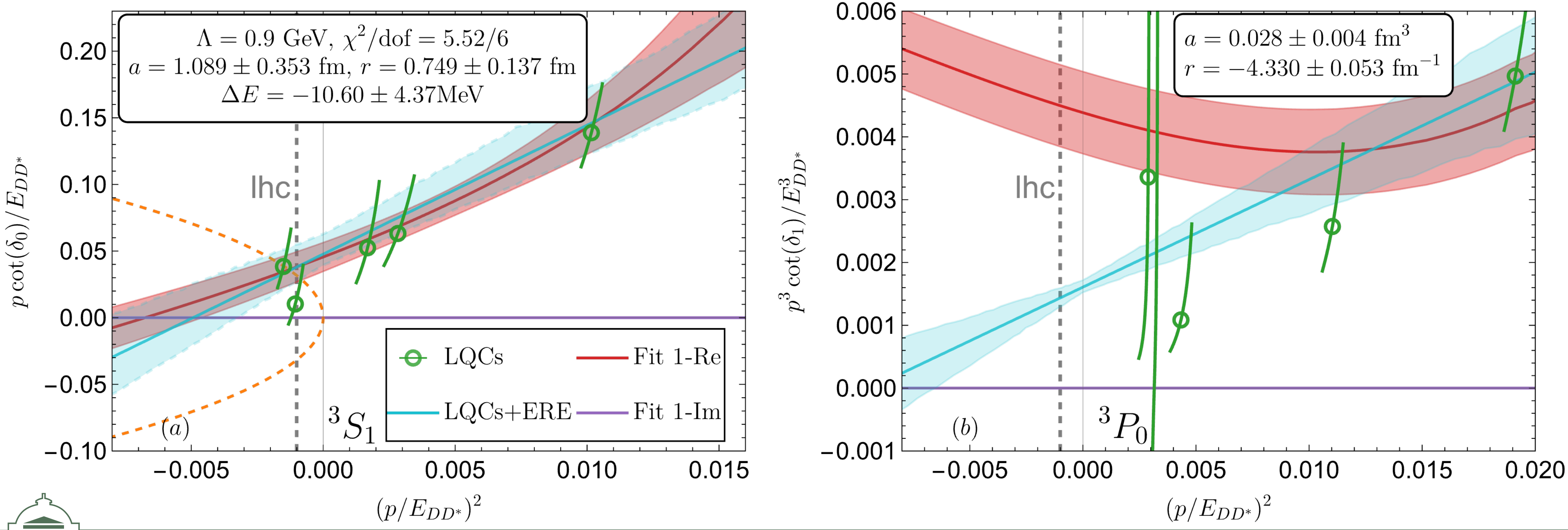
$\chi^2/\text{dof}=3.7/5,\ E^{^3S_1}_{\text{pole}} = -9.9^{+3.6}_{-7.2}$ MeV

$a_{^3S_1} = 1.04(29)\text{fm}, \quad r_{^3S_1} = 0.96^{+0.18}_{-0.20}\text{fm}$

$a_{^3P_0} = 0.076^{+0.008}_{-0.009}\text{fm}^3, \ r_{^3P_0} = 6.9(2.1)\text{fm}^{-1}$

- Hamiltonian + contact terms: 3 paras

  LO and NLO $^3S_1$, NLO $^3P_0$ LECs



$\Lambda = 0.9$ GeV, contact terms+OPE

- LQCs+ ERR: 4 paras.

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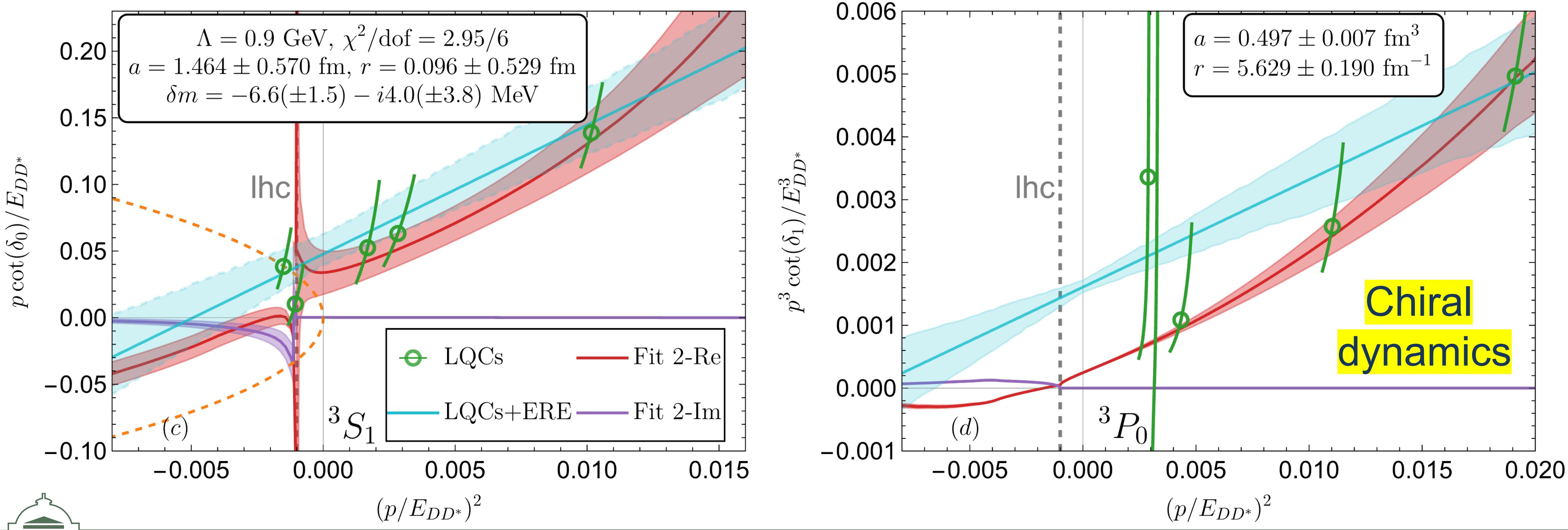
Padmanath:2022cvl

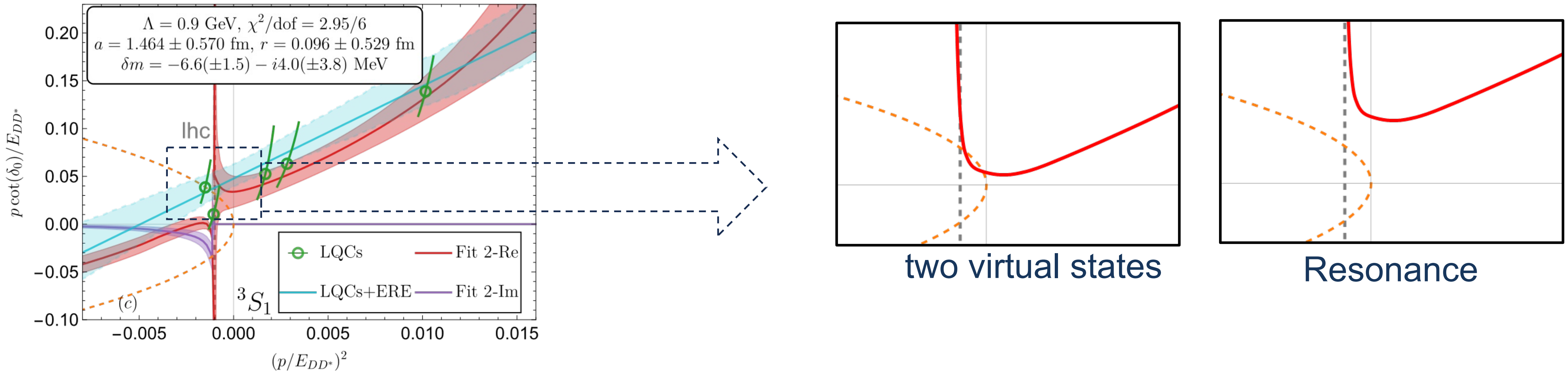
- Hamiltonian + contact terms + OPE: 3 paras

  LO and NLO $^3S_1$, NLO $^3P_0$ LECs



$\Lambda = 0.9$ GeV, contact terms+OPE

- Resonance with 85% probability within the 1σ uncertainty
  - ▶ Rather than the virtual state from Lüscher QC+ERE



Model-(in)dependence

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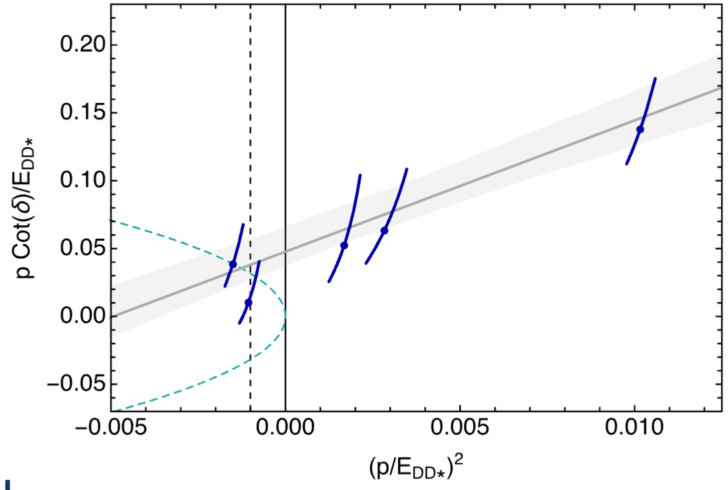
- Question: (modified) Lüscher formula **VS** FV Hamiltonian methods + EFTs

Model-independent *Model-dependent?*

Our answers：

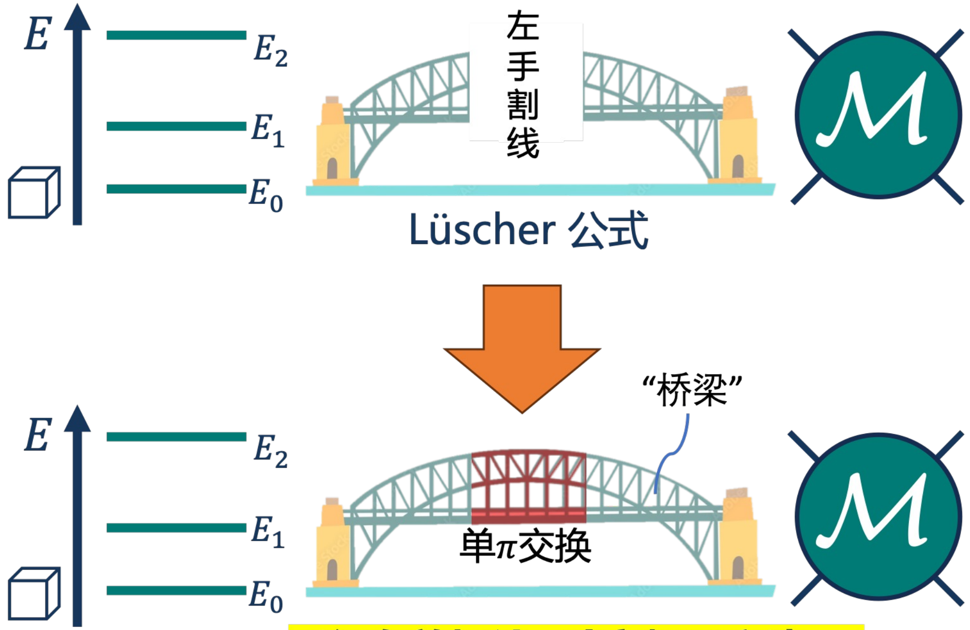
FV Hamiltonian method / EFT **VS** Lüscher QCs / Parametrization of T-matrix



- Parametrization of T-matrix: model dependence
  - ▸ Without PW mixing: one-to-one relation, $E_{FV}\sim\delta_l(E_{FV})$
  - ▸ Phase shift over continue energy: T-matrix parametri.. is needed
  - ▸ PW mixing effect: T-matrix parametri. is needed
- EFTs: also parametri. of the T-matrix with following advantages
  - ▸ Clear breaking down scale
  - ▸ Powering counting, controllable and knowable systemic uncertainties
- FV Hamiltonian method without PW mixing could also provide the one-to-one relation
  - ▸ Define short-range interaction for each E_{FV}^i **separately**, $V_i = \lambda_i V_{short-range}$
  - ▸ Each V_i will give a phase shift $\delta(E_{FV}^i)$ in the infinite volume

Extracting hadronic interaction from lattice QCD raw data

Energy level method



手征有效场论+哈密顿量方法

Potential method

含时的 / 非含时的 → Nambu-Bethe-Salpeter 波函数(原始数据)

微商展开

可分离势展开：EST 方案

→ 势函数 → 观测量

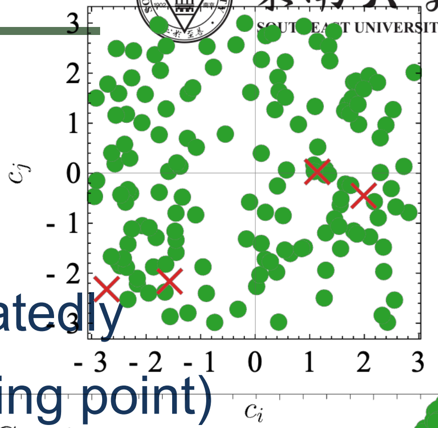
厄米，高效，三体观测量收敛

Thanks for your attention!

BACKUP

- Plane wave basis+Eigenvector continuation
  - ▶ Eigenvector continuation (EC) with subspace learning

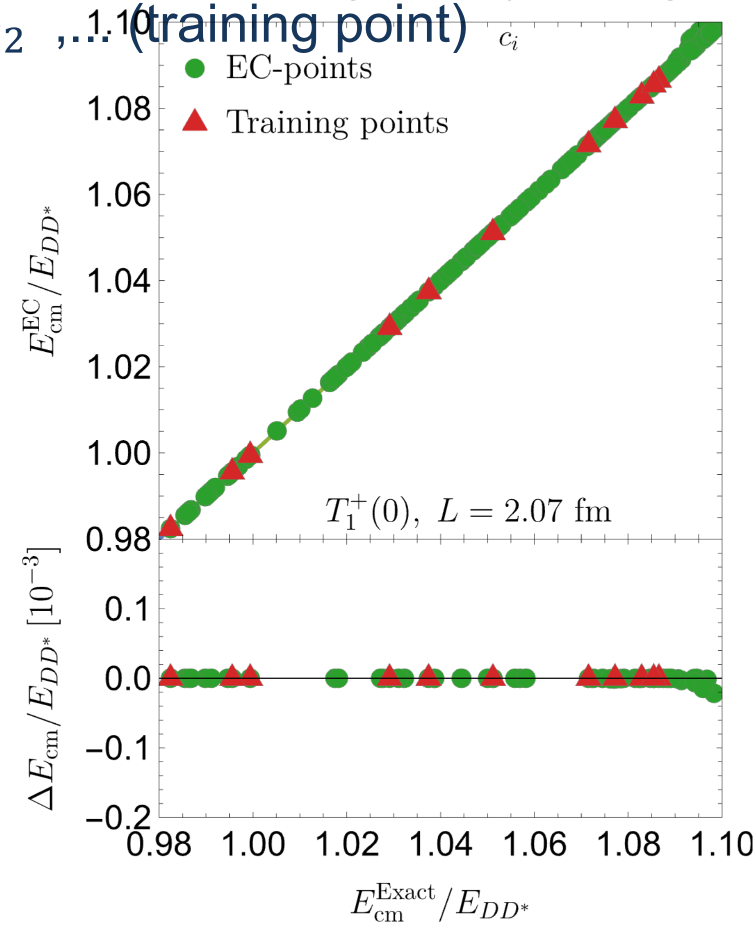
Frame:2017fah,Demol:2019yjt, Furnstahl:2020abp,Yapa:2022nnv



- To fit or quantify uncertainty: solve eigenvalue problem with different $\{c_i\}$ repeatedly
- EC basis: eigenvectors from a selection of parameter sets $\{c_i\}_1$, $\{c_i\}_2$,... (training point)
- Naturalness of LEC in EFT (~ 1) makes the EC more reliable
- dim is linear function

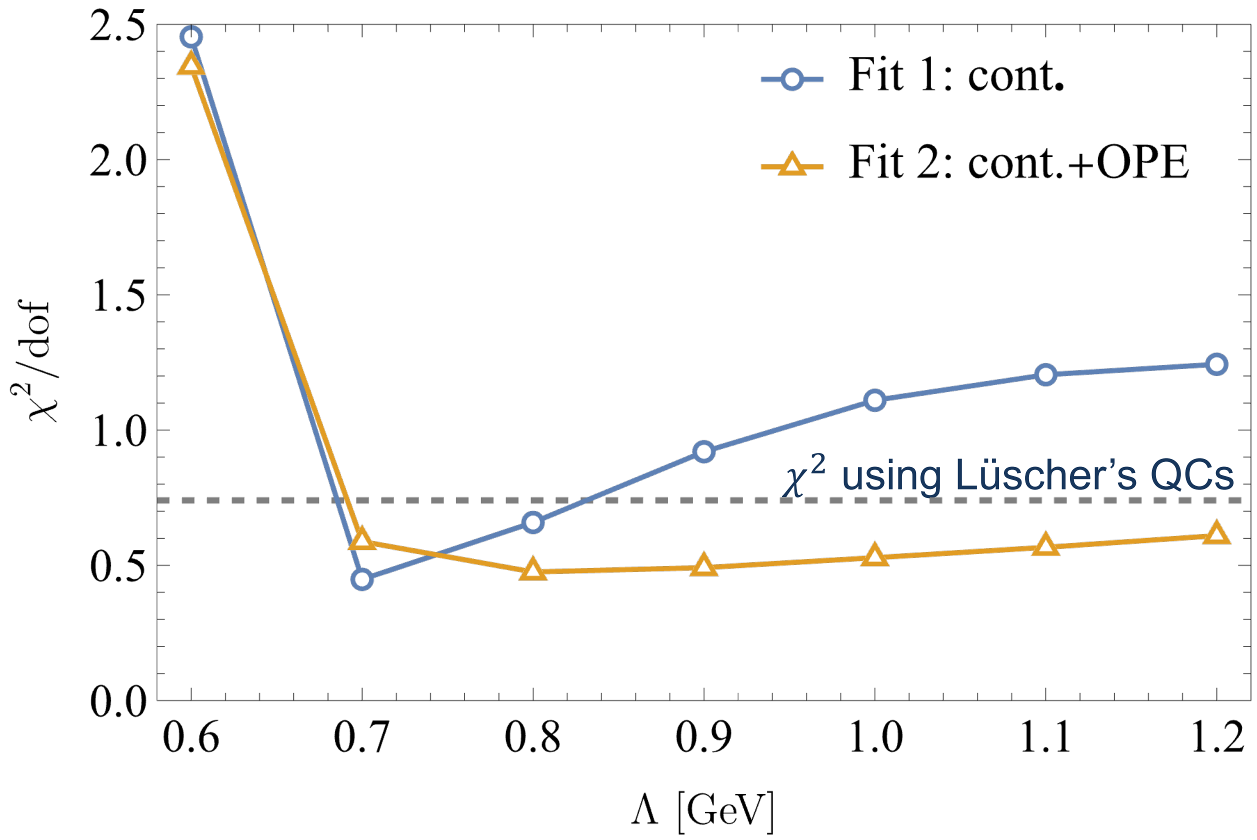
$$\dim^{EC} = \frac{p_{max}}{2\pi/L} \sim \mathcal{O}(10), \; p_{max} \approx 0.6 \text{ GeV}$$

- The subspace learning is the one-time cost
- Make the calculation fast and accurate



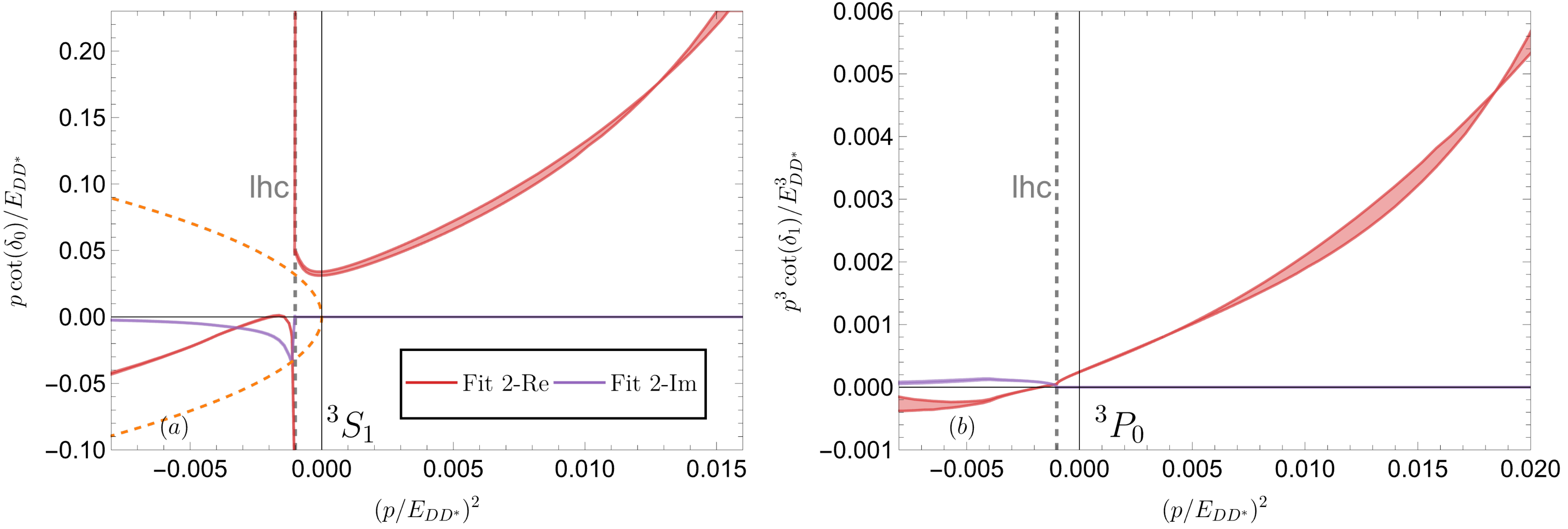
Cutoff dependence of χ^2

- 3 LECs: LO and NLO $^3S_1$ contact terms, NLO $^3P_0$
- In $V_{\rm ctc}$ fit, the P-wave dominate states control Λ-dependence of the χ^2
  - ▶ The shape of the of $k^3 \cot\delta_1$ is determined by regulator and cutoff
  - ▶ Sensitive to Λ
- The $V_{\rm ctc}$+ $V_{1\pi}$ fit is stable with Λ
- The $V_{\rm ctc}$+ $V_{1\pi}$ fit is even better than QCs



Cutoff dependence of phase shift

$\Lambda = 0.7 - 1.2$ GeV



Moving systems

- Moving system in the box $\boldsymbol{P} = \frac{2\pi}{L}\boldsymbol{d} \neq 0$
  - ► For LQCD, changing box size is expensive
  - ► Calculate E^{FV} of moving two-body systems in a box

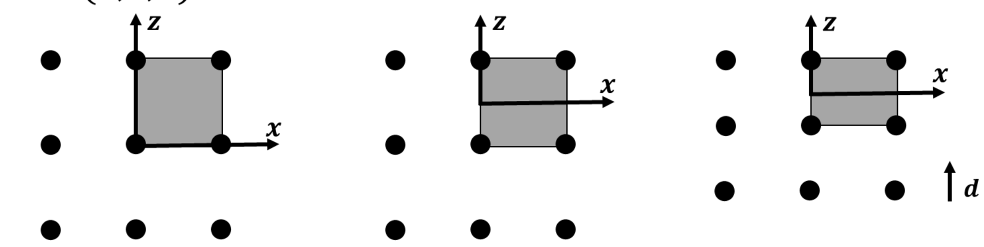
Rummukainen:1995vs,Leskovec:2012gb

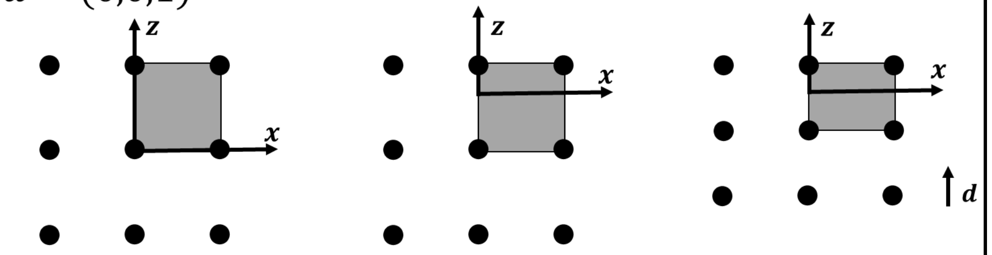
- Box frame (BF) $\boldsymbol{p}$ and center of mass frame (CMF) $\boldsymbol{p}^*$
  - ► BF: $\boldsymbol{p} = \frac{2\pi}{L}\boldsymbol{n}$; CMF: $\boldsymbol{p}^* = \gamma^{-1}\left(\boldsymbol{p}_\parallel - \frac{A}{2}\boldsymbol{P}\right) + \boldsymbol{p}_\perp$
  - ► For moving systems with $m_1 \neq m_2$, states with different parities could mix
- $\boldsymbol{d} = (0,0,1)$, D_{4h} group for $m_1 = m_2$, C_{4v} group for $m_1 \neq m_2$

Space inversion invariance is broken

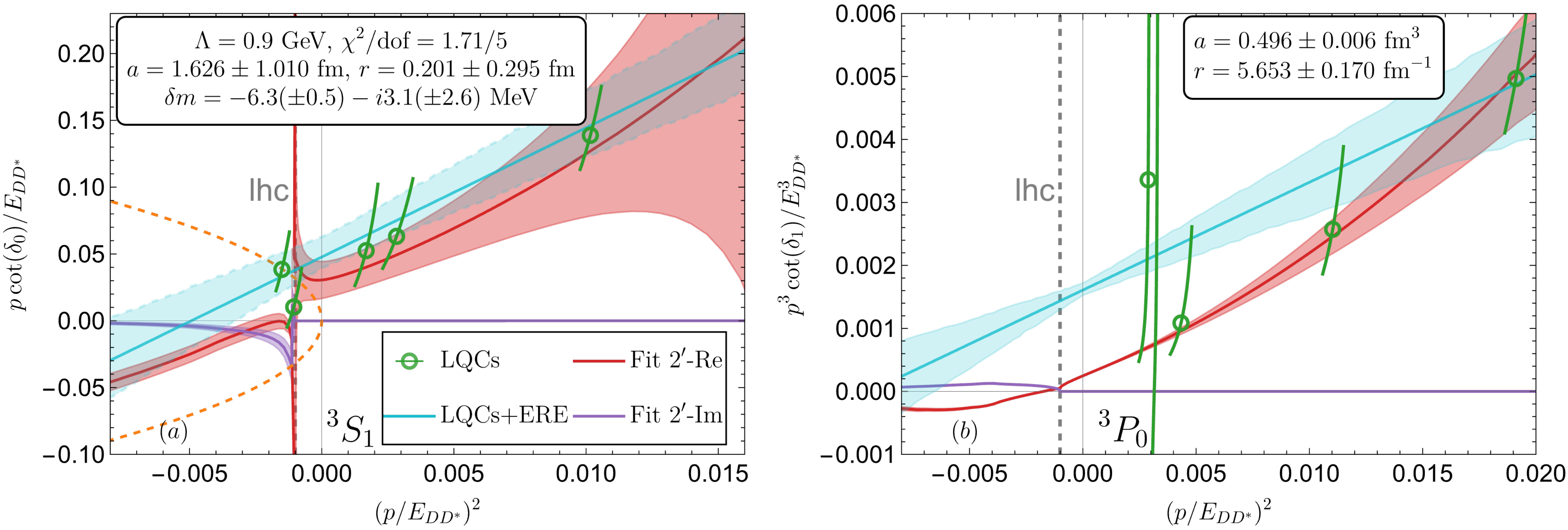
- $\boldsymbol{d} = (1,1,0)$, ...

$m_1 = m_2, \quad A = 1$			$m_1 \neq m_2, \quad A = 1 + \frac{m_1^2 - m_2^2}{E^*}$		
$\boldsymbol{n} \in Z$	$\boldsymbol{n} - \frac{1}{2}\boldsymbol{d}$	$\gamma^{-1}\left(\boldsymbol{n}_\parallel - \frac{\boldsymbol{d}}{2}\right) + \boldsymbol{n}_\perp$	$\boldsymbol{n} \in Z$	$\boldsymbol{n} - \frac{A}{2}\boldsymbol{d}$	$\gamma^{-1}\left(\boldsymbol{n}_\parallel - \frac{A}{2}\boldsymbol{d}\right) + \boldsymbol{n}_\perp$
$\boldsymbol{d} = (0,0,1)$			$\boldsymbol{d} = (0,0,1)$		

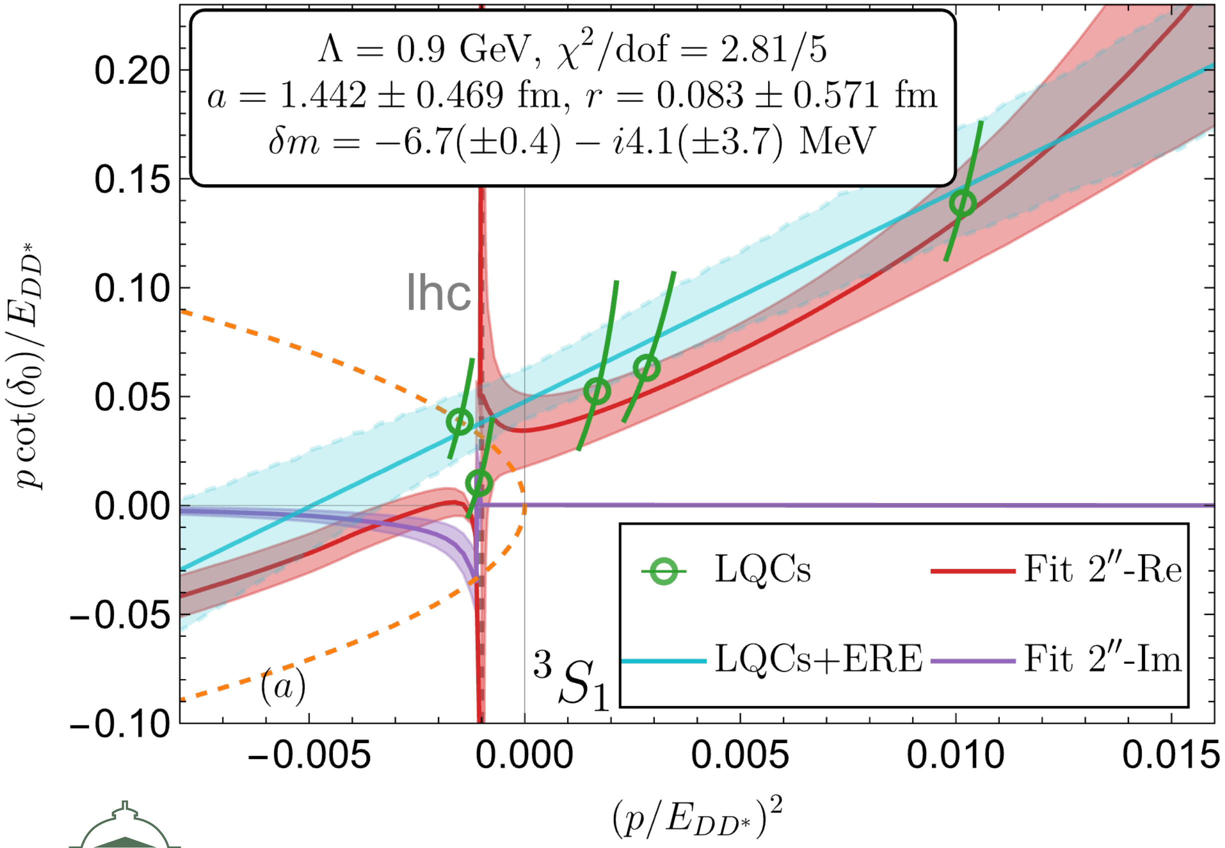


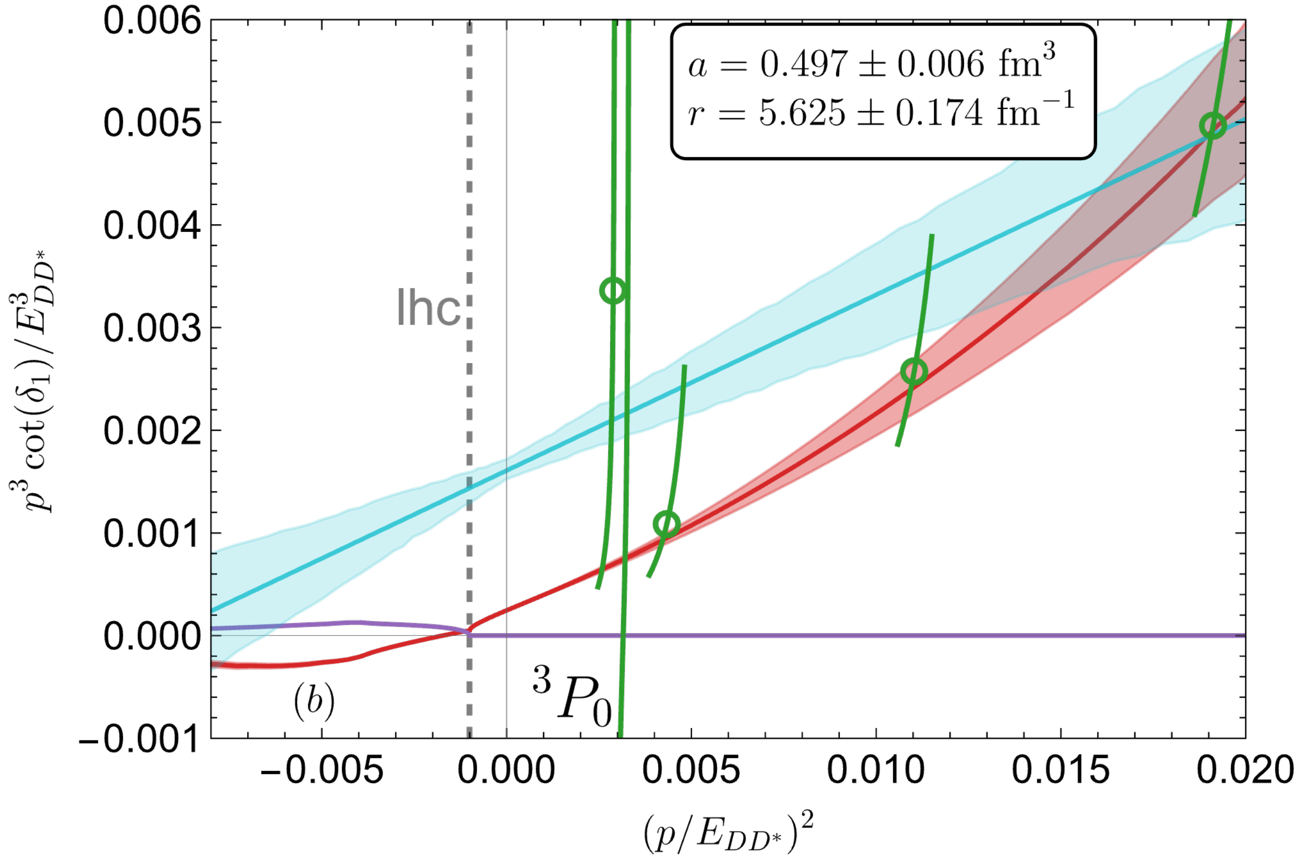


Including SD transition terms



Including 3P2 term





Hamiltonian approach in Plane wave basis: $|p_n, \eta\rangle$

- Seven patterns of representation space $\{n_1, n_2, n_3\}_{dim}$ for O_h group

$$\Rightarrow \{0,0,0\}_{1\times3}, \{0,0,a\}_{6\times3}, \{0,a,a\}_{12\times3}, \{0,a,b\}_{24\times3}\dots$$

- Reduce to irreducible representations (irreps): projection operator

e.g. textbook by M.Dresselhaus et.al

$$\hat{P}^{\Gamma_a}_{\alpha\beta} \equiv \sum_{g_i\in G} \frac{N(\Gamma_a)}{n_G} R^{\Gamma_a}_{\alpha\beta}(g_i)^* \hat{D}(g_i), \quad \hat{P}^{\Gamma_a}_{\alpha\alpha'}|\psi\rangle = a^{\Gamma_a}_{\alpha'}|\Gamma_a, \alpha\rangle.$$

- An example: $\{0,0,a\}_{6\times3} = 2T_1^+ \oplus T_2^+ \oplus A_1^- \oplus E_1^- \oplus T_1^- \oplus T_2^-$
- For moving systems, elongated boxes, particles with arbitrary spin...

Symmetric group (character table) $\xrightarrow{\hat{P}^{\Gamma}}$ unitary irrep matrices $\xrightarrow{\hat{P}^{\Gamma}_{\alpha\beta}}$ rep space $|p_n\rangle \rightarrow$ irreps

- dim of the $\mathbb{H}_\Gamma$: cubic function of L^{-1}

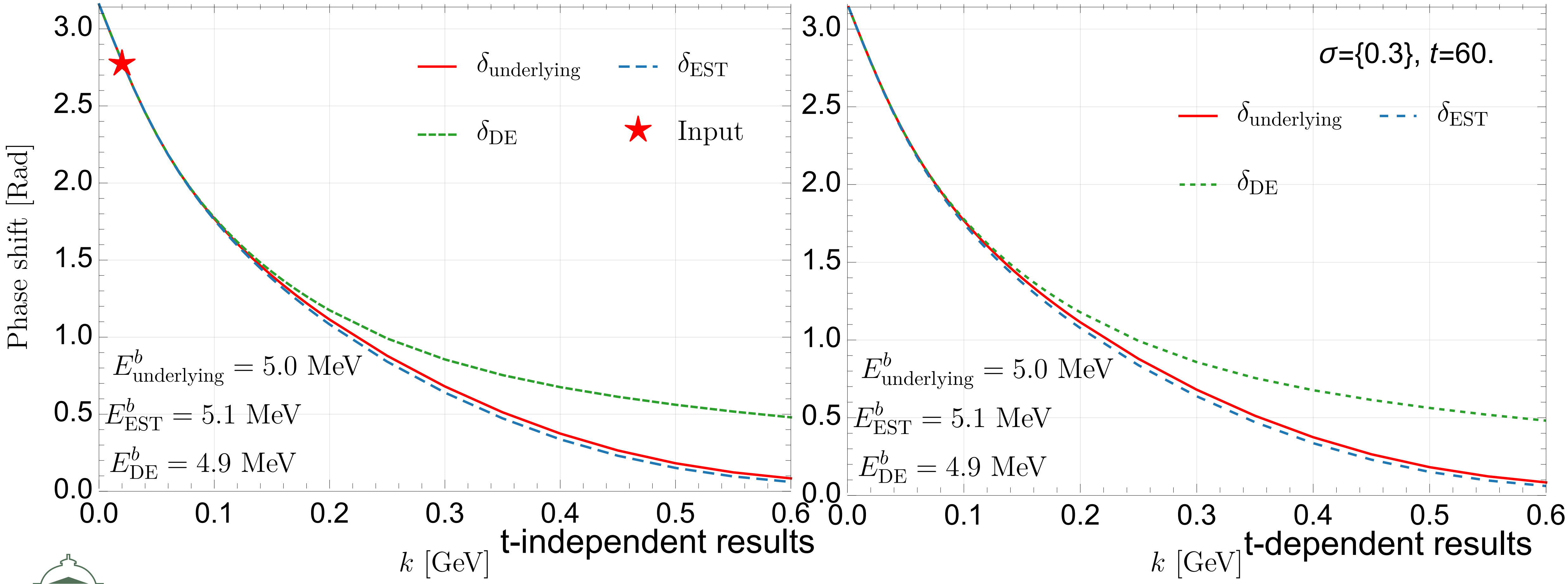
$$\dim \sim \left(\frac{\Lambda_{\rm UV}}{2\pi/L}\right)^3 \times \frac{1}{10} \sim \mathcal{O}(1000)$$

Bound state

- At LO, both EST and DE method give reasonable binding energy
- The EST method perform better in phase shift
- Singular potential in DE at NLO

$$V_{ctc}(\boldsymbol{p},\boldsymbol{p}') = Ce^{-\frac{p^2+p'^2}{\Lambda^2}},$$

$$V_{ope}(\boldsymbol{q}) = -\frac{g_A}{4F_\pi^2}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{q}\boldsymbol{\sigma}_2\cdot\boldsymbol{q}}{\boldsymbol{q}^2+m_\pi^2} + C_{sub}\boldsymbol{\sigma_1}\cdot\boldsymbol{\sigma_2}\right)e^{-\frac{q^2+m_\pi^2}{\Lambda^2}}$$



t-independent results

t-dependent results

Bound state

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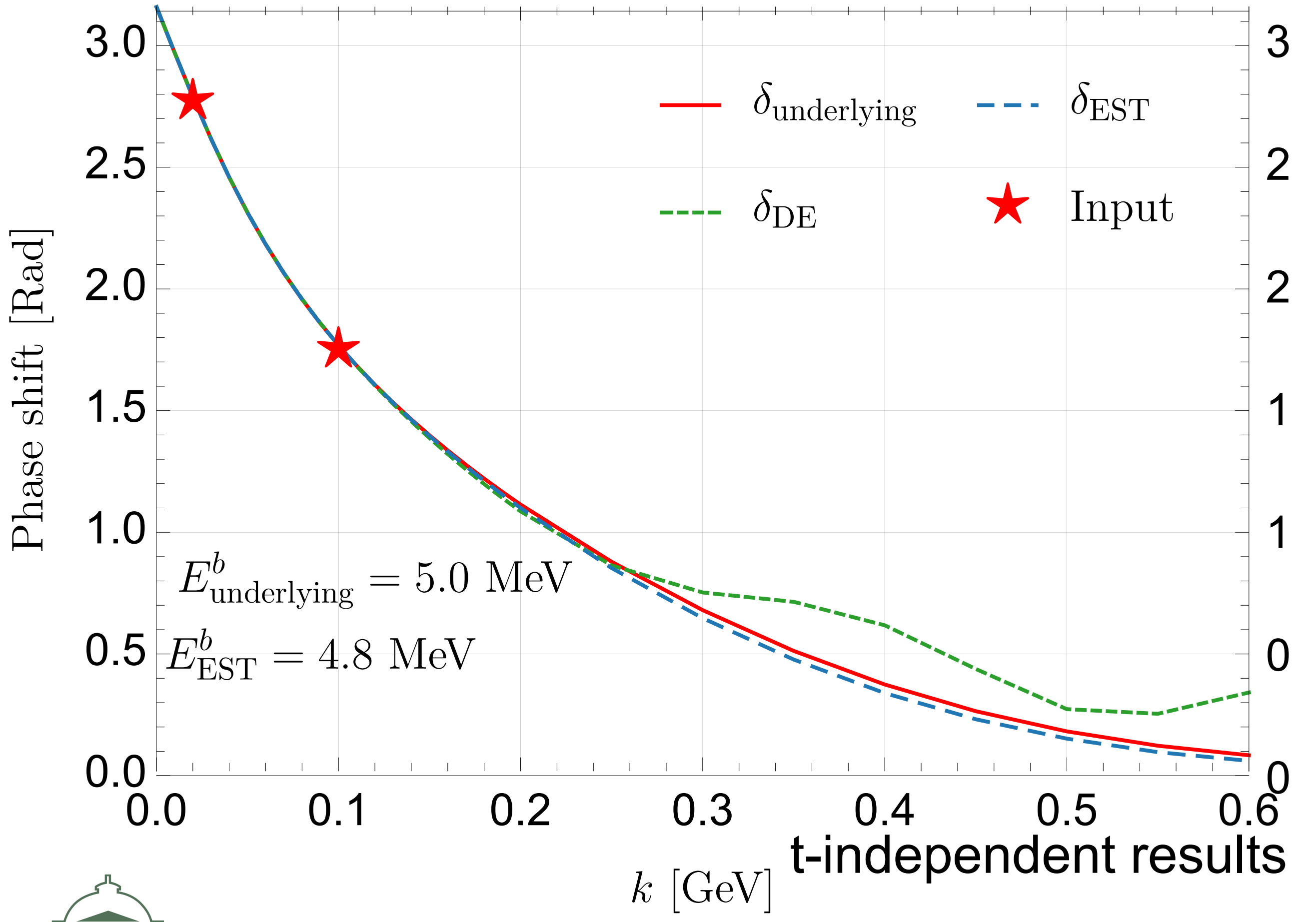
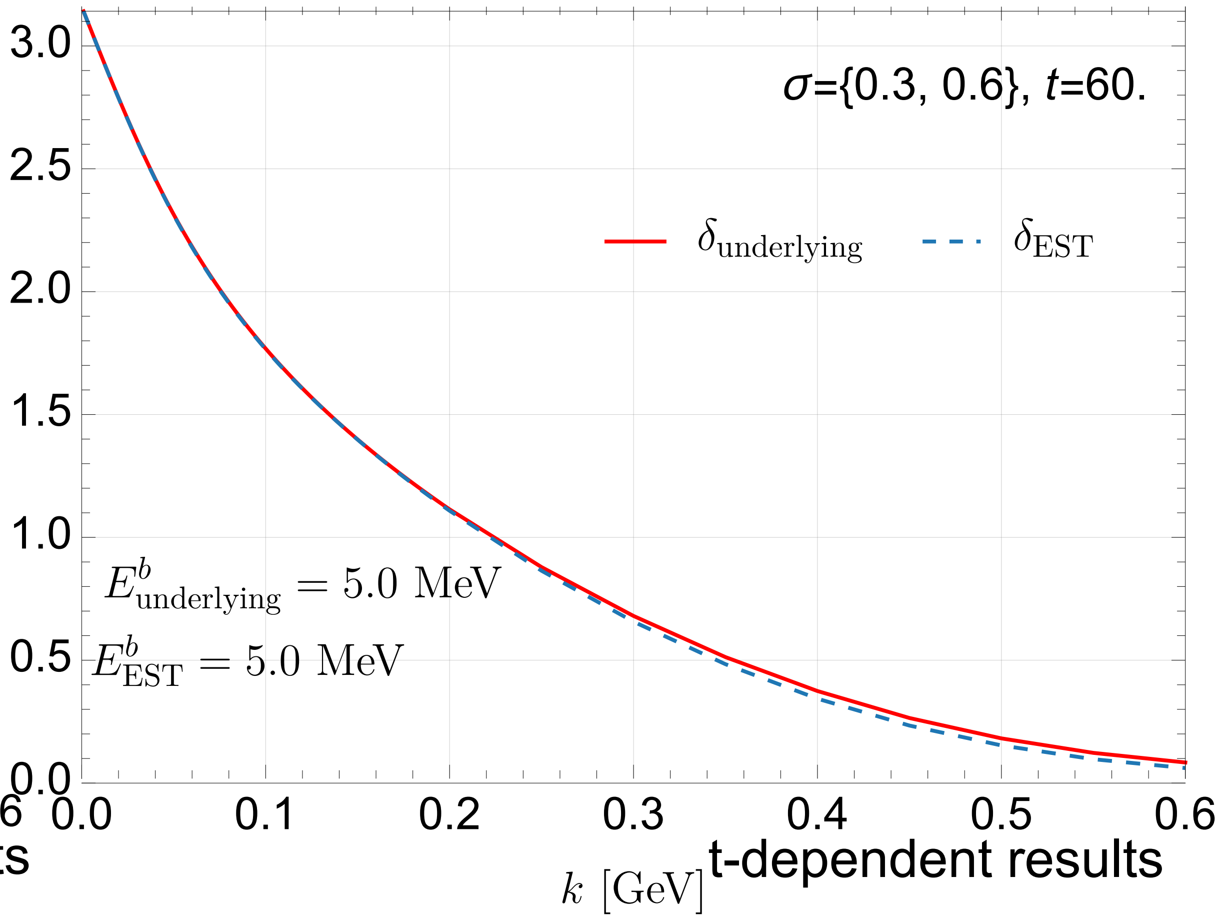
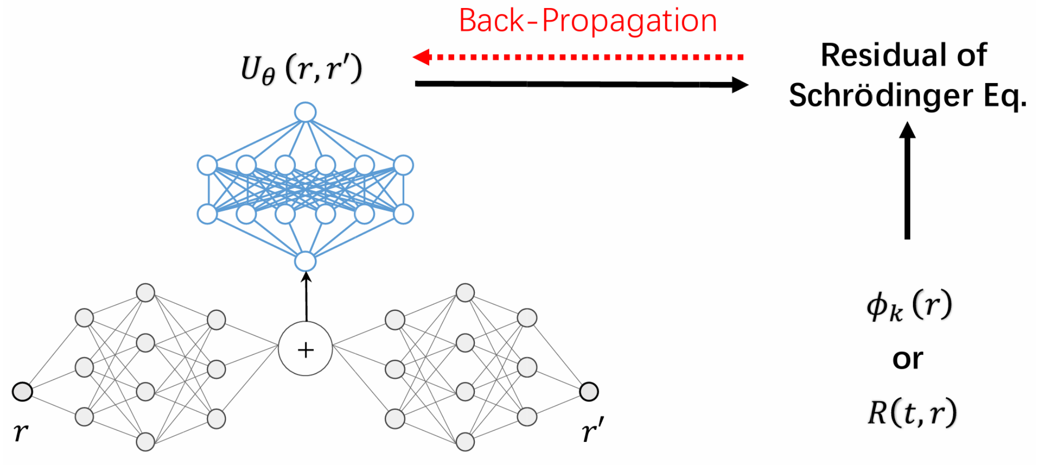


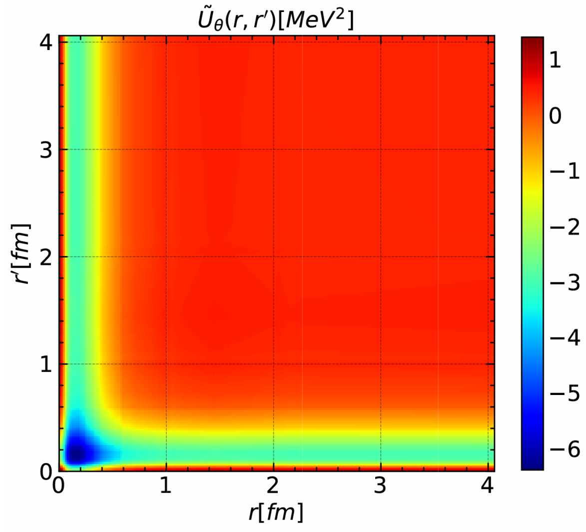
t-independent results



t-dependent results

Machine learning offer a solution?





Wang, L., Doi, T., Hatsuda, T., & Lyu, Y. (2025). *PoS*, *LATTICE2024*, 076

- Deep neutral network as the representation of potential
  - ▶ General potential function: nonlocal potential
- Cannot alter the fact that the solution is not unique

$$\int dr' V(r, r') R^{(i)}(r) = K^{(i)}(r) \Rightarrow \mathbb{V}_{N\times N} R^{(i)}_{N\times 1} = K^{(i)}_{N\times 1}$$

N: # of quadrature points
N wave functions to fix potential matrix $\mathbb{V}_{N\times N}$

- Promising approach the systemic uncertainties of the different potential parameterization

discrepancy

ceive significant contributions from excited states. For single hadrons, the excited state gap is roughly set by $2m_\pi$. Thus, the ground state is resolvable over a Euclidean time of roughly $1-2$ fm. In contrast, for NN calculations, the excited state gap is roughly set by the quantized momentum modes of the nucleon. With periodic spatial boundary conditions, these are $\Delta E_n = 2\sqrt{m_N^2+p_n^2} - 2m_N \approx p_n^2/m_N$ with $p_n = \sqrt{n}(2\pi/L)$ for integer n. For typical sizes of the lattice L, these energy gaps are $O(20-50)$ MeV and thus, without a suppression of the excited states, the ground state is not resolvable until $4-10$ fm in time while the S/N for NN calculations has degraded around $t \sim 2$ fm. Thus, for correlation func-

This discrepancy was believed to be due to uncontrolled systematic uncertainties with the HAL QCD Potential method [19–24]. At the same time, subsequent work by HAL QCD raised serious concerns about the calculations that observed bound states [16–18, 25]. This was followed up by two-baryon calculations that utilized a set of operators that enabled the use of momentum space creation operators, in contrast to the local hexa-quark (HX) creation operators used in Ref. [9–15] (these operators will be defined subsequently). These new methods found significantly reduced binding energies in the case of the h-dibaryon [26, 27], or a lack of bound states in the case of di-nucleons [28, 29]. This discrepancy has long plagued progress toward the physical point and led to a lack of confidence in results reported for further two- and higher- nucleon quantities [30, 31].

Di-nucleons do not form bound states at heavy pion mass

BaSc Collaboration • John Bulava (Ruhr U., Bochum) Show All(19)

May 8, 2025

30 pages

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View in: ADS Abstract Service

Time-independent HAL QCD

- The equal-time BS amplitude (BS wave function, BSWF)

CP-PACS:2005gzm

$$\psi(\vec{x};\vec{k}) = \langle 0|\pi_1(\vec{x}/2)\pi_2(-\vec{x}/2)|\pi_1(\vec{k}),\pi_2(-\vec{k});in\rangle$$

- Asymptotic behavior of BS wave function

$$\psi(\vec{x};\vec{k}) = e^{i\vec{k}\cdot\vec{x}} + \int \frac{d^3p}{(2\pi)^3}\frac{T(p;k)}{p^2-k^2-i\epsilon}e^{i\vec{p}\cdot\vec{x}}$$

  - ► $T(p;k)$ is the half-on-shell T-matrix
  - ► $\psi(\vec{x};\vec{k})$ satisfy the Lippmann-Schwinger eq. as the non-relativistic scattering wave function
  - ► Using Nishijima, Zimmermann and Haag (NZH) reduction formula for composite local operators
  - ► No non-relativistic approximation, just a formal resemblance

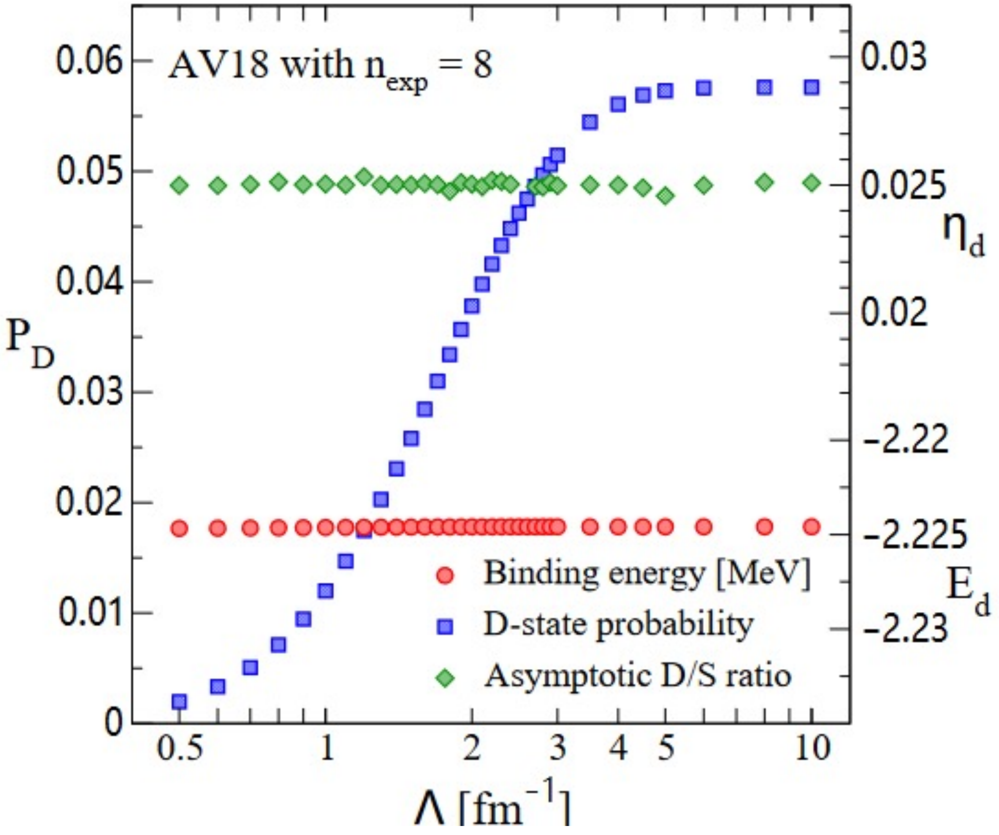
- The BSWF at different energies $\{k_i\}$ in the lattice are the raw data of t-independent HAL QCD

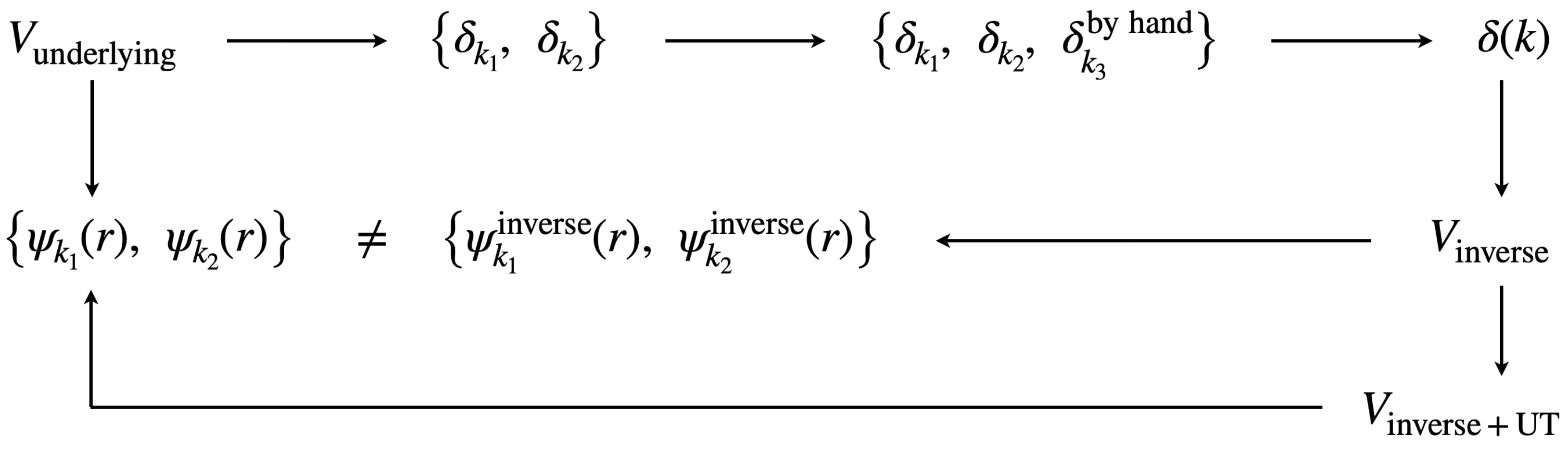
- The general problem: $\psi_{k_i}(\vec{x}) \Rightarrow V$ from Schrodinger-like equations

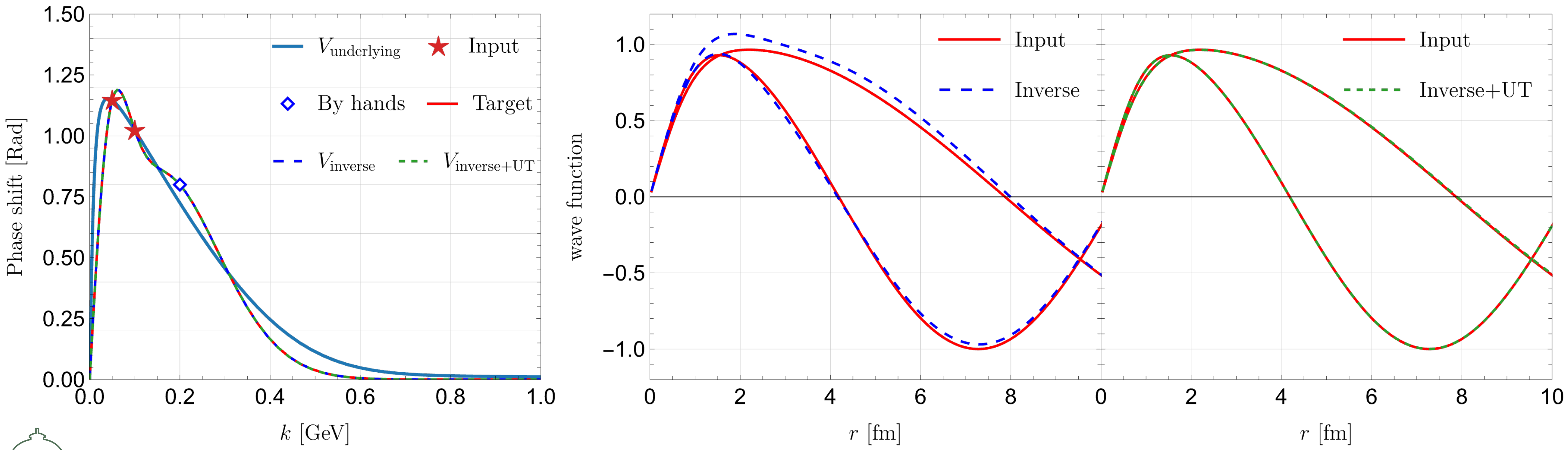
Observable or not?

- Non-observables
  - ▶ Non-asymptotic behavior of ψ, e.g. the deuteron D-state probability
  - ▶ Off-shell T-matrix
  - ▶ Potential
  - ▶ Source functions.
- Observables
  - ▶ Asymptotic behavior of ψ
  - ▶ Phase shift
  - ▶ On-shell T-matrix
  - ▶ Correlation functions

Amghar:1995av



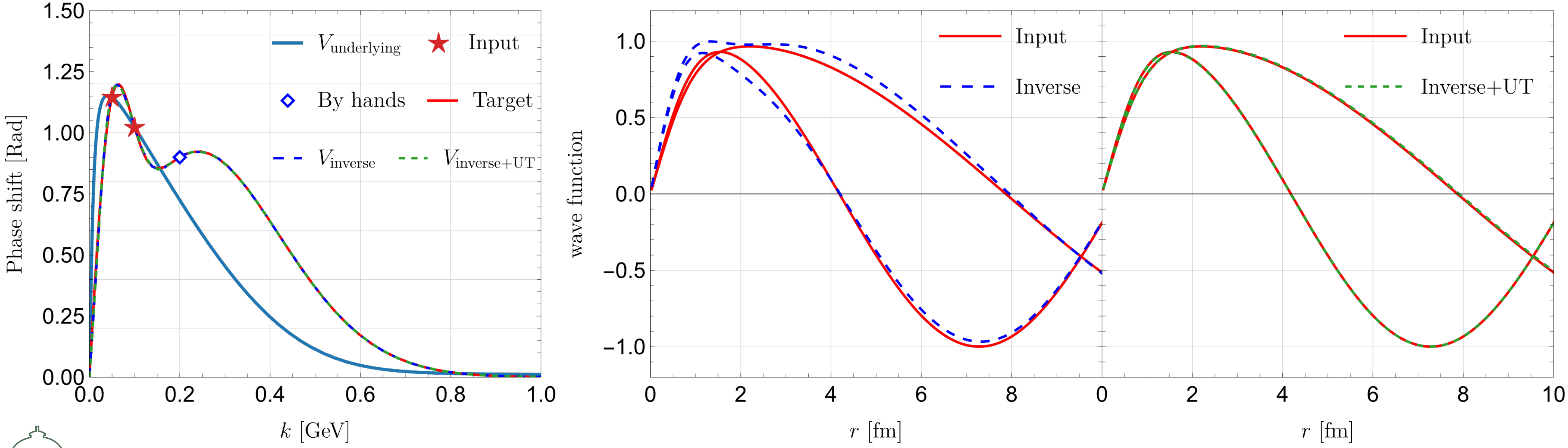

$V_{\text{underlying}}$
$\{\delta_{k_1},\ \delta_{k_2}\}$
$\{\delta_{k_1},\ \delta_{k_2},\ \delta_{k_3}^{\text{by hand}}\}$
$\delta(k)$
$\{\psi_{k_1}(r),\ \psi_{k_2}(r)\} \neq \{\psi_{k_1}^{\text{inverse}}(r),\ \psi_{k_2}^{\text{inverse}}(r)\}$
V_{inverse}
$V_{\text{inverse}+\text{UT}}$

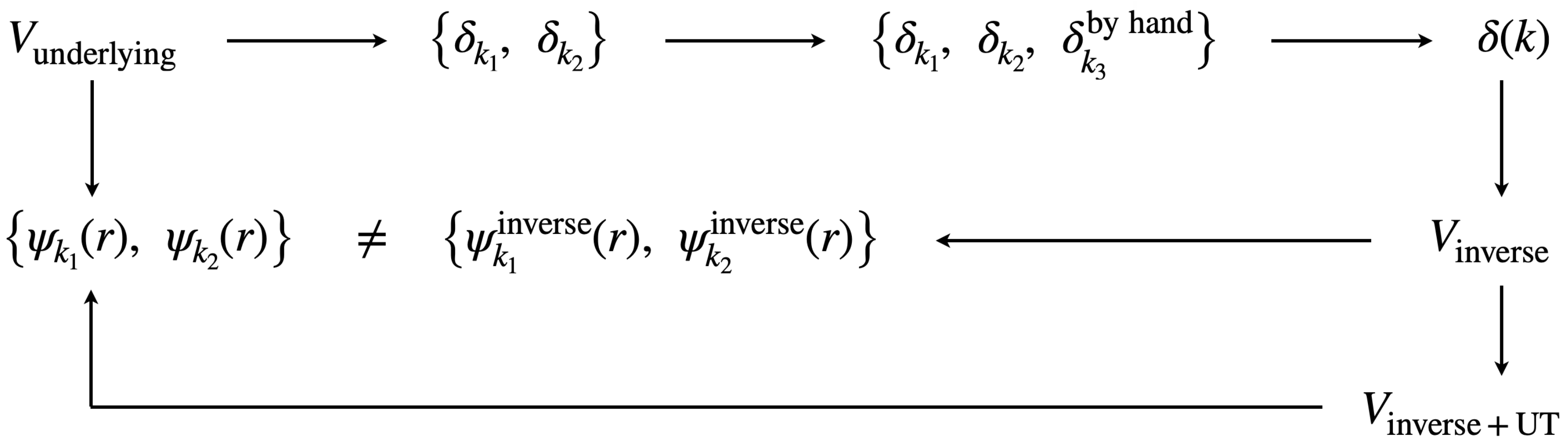

Phase shift [Rad]
k [GeV]
$V_{\text{underlying}}$
Input
By hands
Target
V_{inverse}
$V_{\text{inverse+UT}}$
wave function
r [fm]
Input
Inverse
Input
Inverse+UT

$$V_{\text{underlying}} \longrightarrow \{\delta_{k_1},\ \delta_{k_2}\} \longrightarrow \{\delta_{k_1},\ \delta_{k_2},\ \delta_{k_3}^{\text{by hand}}\} \longrightarrow \delta(k)$$

$$\{\psi_{k_1}(r),\ \psi_{k_2}(r)\} \neq \{\psi_{k_1}^{\text{inverse}}(r),\ \psi_{k_2}^{\text{inverse}}(r)\} \longleftarrow V_{\text{inverse}}$$

$$V_{\text{inverse}+\text{UT}}$$




$V_{\text{underlying}}$
$\{\delta_{k_1},\ \delta_{k_2}\}$
$\{\delta_{k_1},\ \delta_{k_2},\ \delta_{k_3}^{\text{by hand}}\}$
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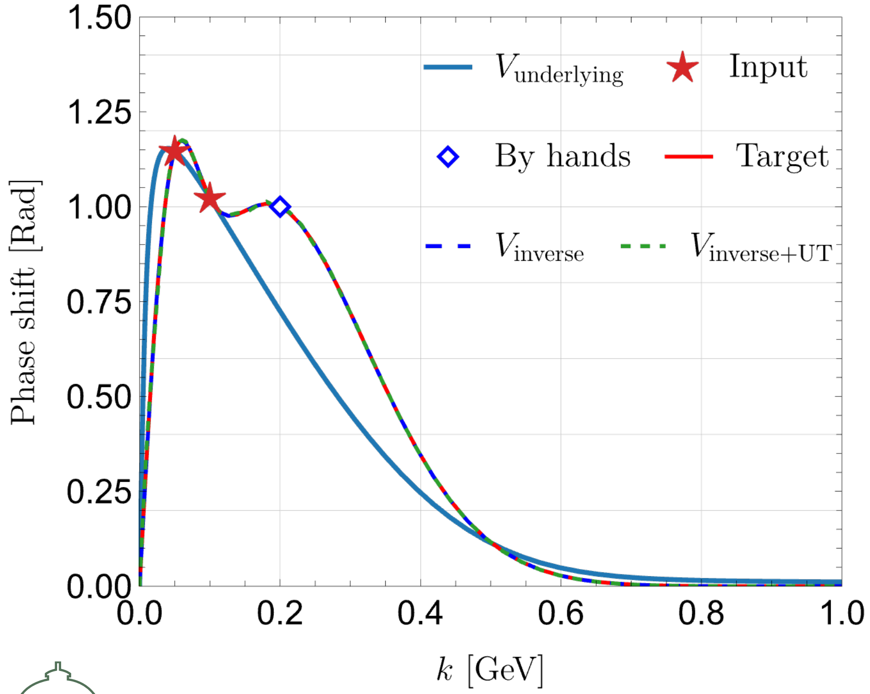
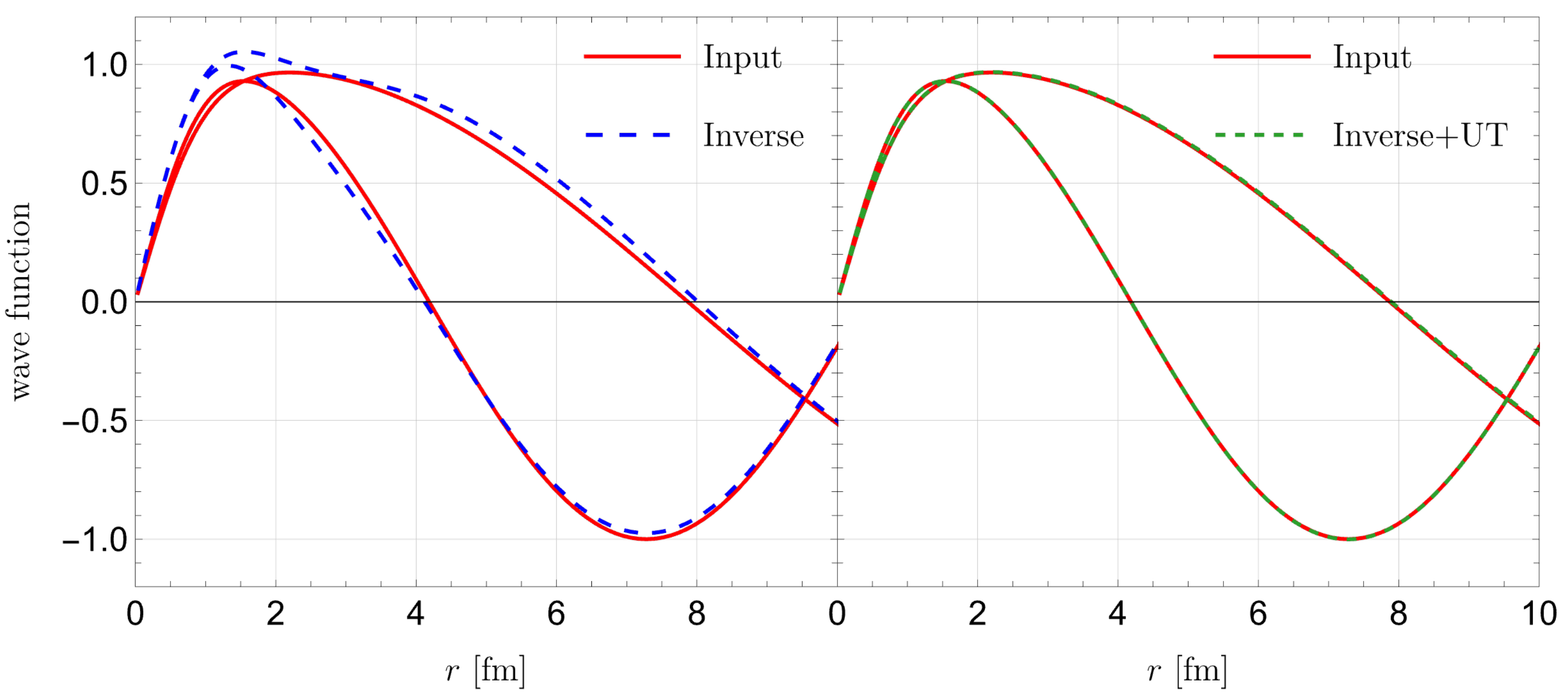

Phase shift [Rad]
k [GeV]
$V_{\text{underlying}}$
Input
By hands
Target
V_{inverse}
$V_{\text{inverse+UT}}$


wave function
r [fm]
Input
Inverse
Input
Inverse+UT

Some tips

Two different expansion

- using the potential to perform formal expansion
    - ► A regular potential could be singular after expansion
- Using the potential generate the wave function and then use the wave function to generate potential

- Why only even term

odd derivative terms are absent in the hermitian potential with rotational and time-reversal symmetries, we do not need such terms to describe scattering phase shift.

Proof of hermitian

$$\hat{V}|R^{(i)}\rangle = K^{(i)}$$

$$K^{(i)} = (E - \hat{H}_0)R^{(i)}$$

$$\hat{V} = \sum_{mn} |K^{(m)}\rangle\Lambda_{mn}\langle K^{(n)}|, \quad \sum_{n} \Lambda_{mn}\langle K^{(n)}|R^{(i)}\rangle = \delta_{mi}$$

Hermitian

$$\hat{V}^\dagger = \sum_{mn} |K^{(m)}\rangle\Lambda^*_{nm}\langle K^{(n)}|$$

the matrix $\{\langle K^{(n)}|R^{(i)}\rangle\}$ is Hermitian and so it is $\{\Lambda_{mn}\}$

$$\begin{aligned}\langle K^{(m)}|R^{(n)}\rangle &= \langle (E - \hat{H}_0)R^{(m)}|R^{(n)}\rangle \\ &= \langle R^{(m)}|E - \hat{H}_0|R^{(n)}\rangle\end{aligned}$$