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Tetraquark bound states in constituent quark models: Benchmark calculations

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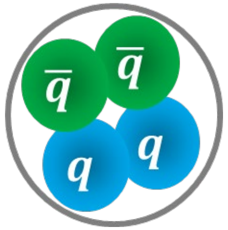
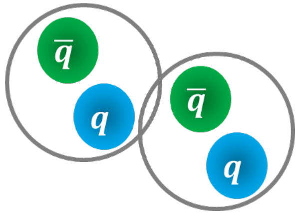
8th Jan., 2024

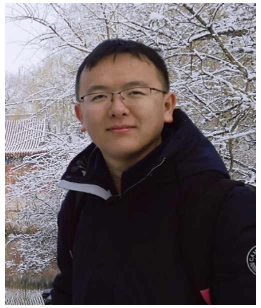


Based on PRD108 (2023)114016, PRD107(2023)054035, 2309.17068, 2310.14597.

Together with Yan-Ke Chen, Yao Ma and Shi-Lin Zhu (PKU)



- Background
- Quark interactions
- Gaussian expansion method
  - ▶ Works the best
- Resonating group method
- Diffusion monte Carlo method
- Summary

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Background

History of the multiquark states

Phys.Lett. 8 (1964) 214-215

Volume 8, number 3 PHYSICS LETTERS 1 February 1964

A SCHEMATIC MODEL OF BARYONS AND MESONS *

M. GELL-MANN
California Institute of Technology, Pasadena, California

Received 4 January 1964

...

A simpler and more elegant scheme can be constructed if we allow non-integral values for the charges. We can dispense entirely with the basic baryon b if we assign to the triplet t the following properties: spin $\frac{1}{2}$, $z = -\frac{1}{3}$, and baryon number $\frac{1}{3}$. We then refer to the members $u^{\frac{2}{3}}$, $d^{-\frac{1}{3}}$, and $s^{-\frac{1}{3}}$ of the triplet as "quarks" 6) q and the members of the anti-triplet as anti-quarks $\bar{q}$. Baryons can now be constructed from quarks by using the combinations (qqq), $(qqqq\bar{q})$, etc., while mesons are made out of $(q\bar{q})$, $(qq\bar{q}\bar{q})$, etc. It is assuming that the lowest baryon configuration (qqq) gives just the represen-

8419/TH.412
21 February 1964

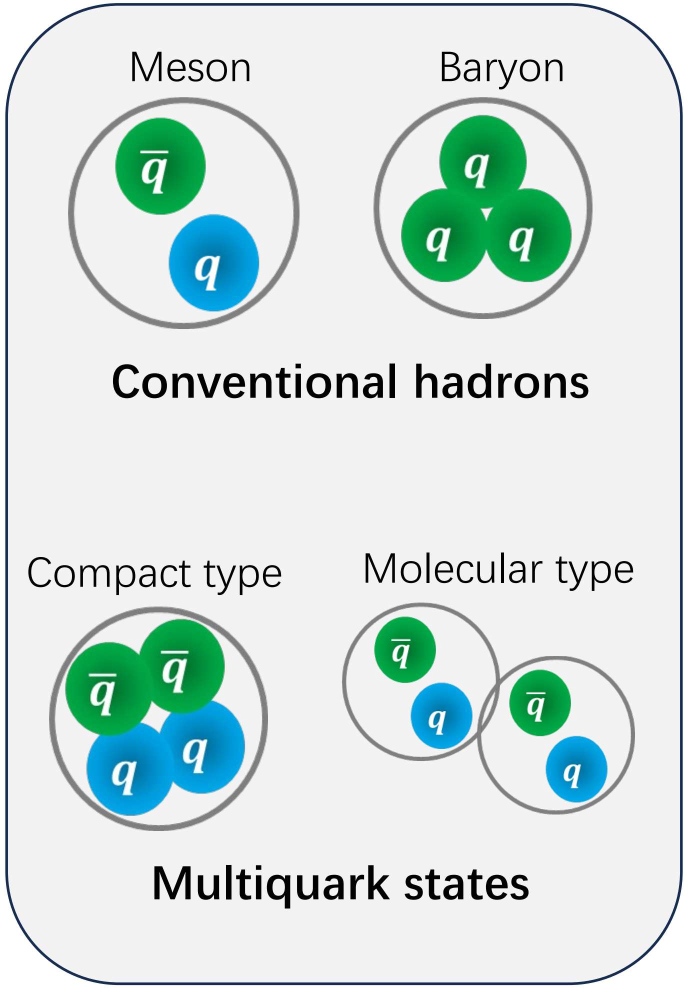
AN SU_3 MODEL FOR STRONG INTERACTION SYMMETRY AND ITS BREAKING
II *)

G. Zweig
CERN---Geneva

*) Version I is CERN preprint 8182/TH.401, Jan. 17, 1964.

...

6) In general, we would expect that baryons are built not only from the product of three aces, AAA, but also from $\bar{A}AAAA$, $\bar{A}\bar{A}AAAAA$, etc., where $\bar{A}$ denotes an anti-ace. Similarly, mesons could be formed from $\bar{A}A$, $\bar{A}\bar{A}AA$ etc. For the low mass mesons and baryons we will assume the simplest possibilities, $\bar{A}A$ and AAA, that is, "deuces and treys".

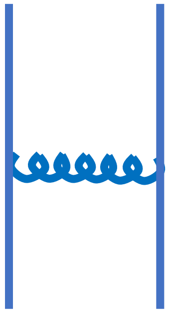


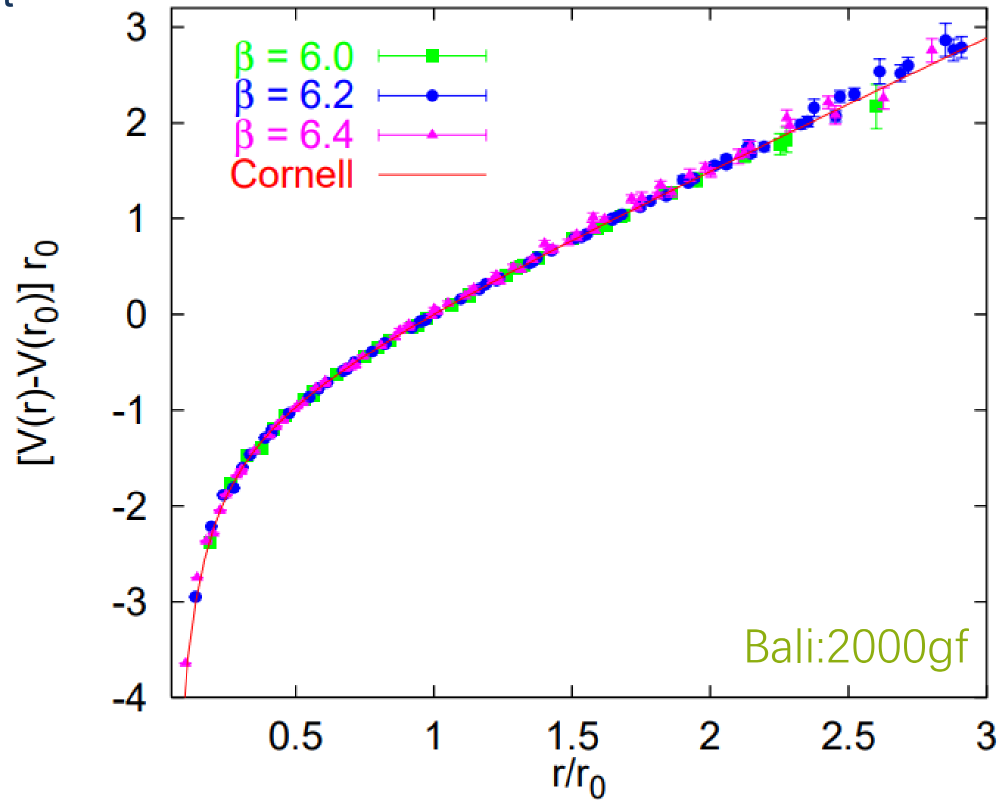
- The multiquark states were predicted at the birth of Quark model

Quark potential models

- A minimal model: one-gluon-exchange+Confinement   Eichten:1978tg, Barnes:2005pb⋯

$$V_{ij}(r) = \Big[\underbrace{\frac{\alpha_s}{r} - \frac{8\pi\alpha_s}{3m_i m_j}\frac{\tau^3}{\pi^{3/2}}e^{-\tau^2 r^2}\boldsymbol{s}_i\cdot\boldsymbol{s}_j}_{\text{OGE}} + \underbrace{\Big(-\frac{3b}{4}r + V_c\Big)}_{\text{Confinement}}\Big]\frac{\lambda_i\cdot\lambda_j}{4}$$

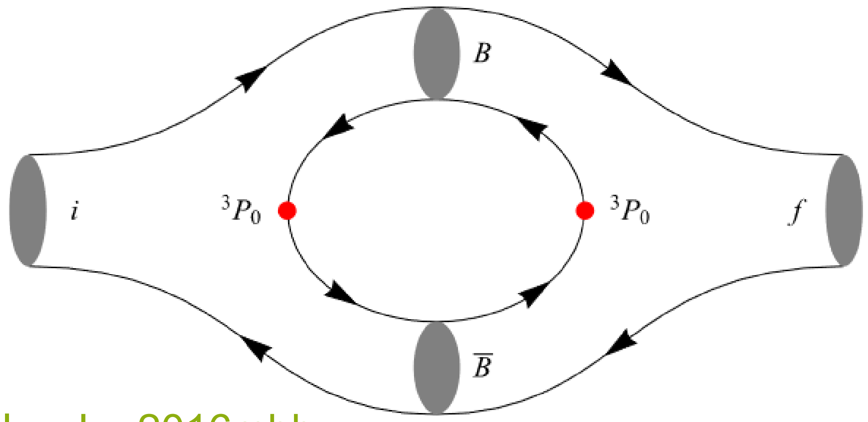




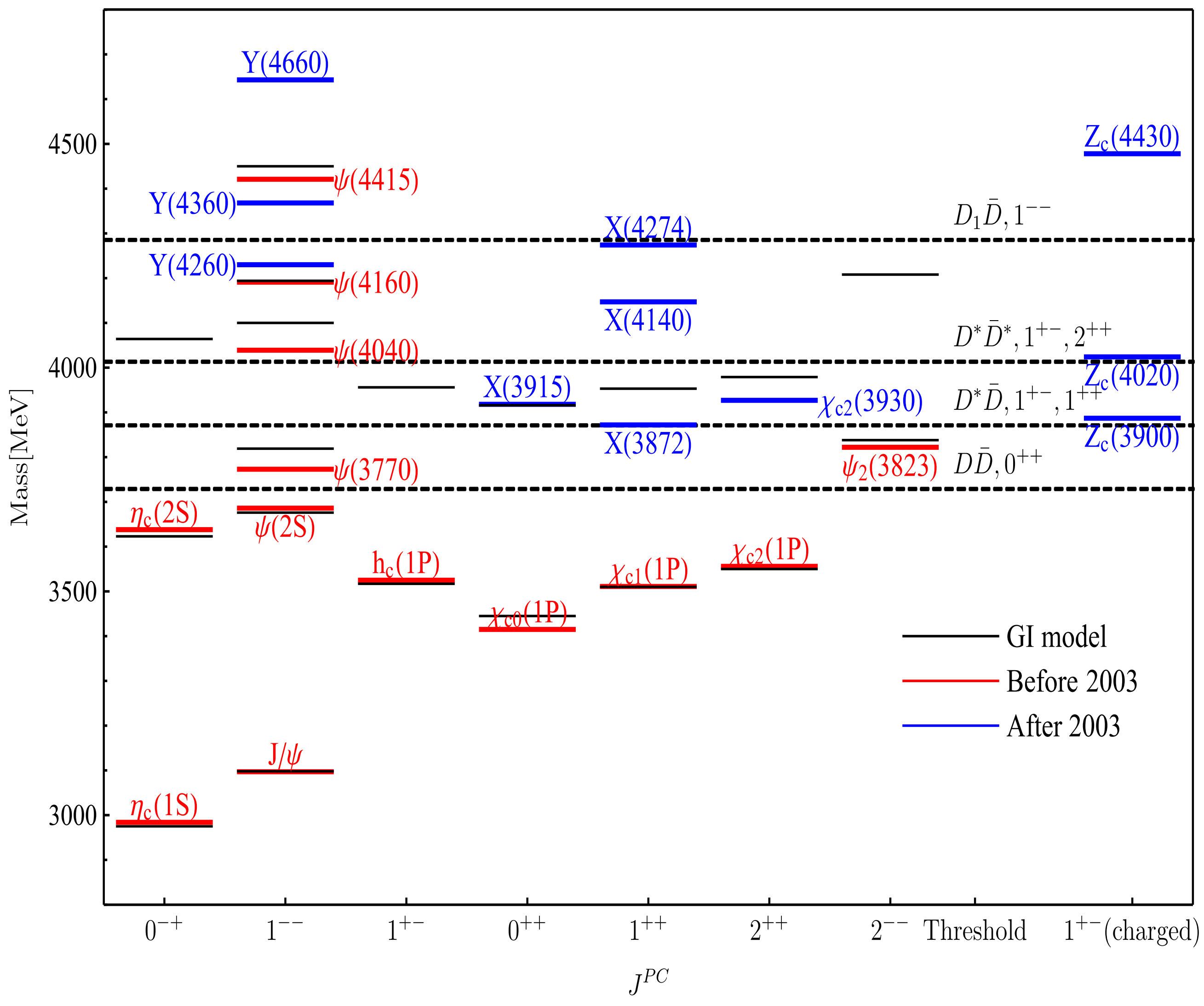
- Relativized Godfrey-Isgur model   Godfrey:1985xj

Tetraquark states

- Since 2003, many heavy-quarkonium-like states were observed.
  - ▶ Hard to include them in the pattern predicted by quark models
  - ▶ Most of them are above the di-meson thresholds
- The unquenched effect



Ni:2023lvx, Lu:2016mbb,…



- Ones with exotic quantum numbers are tetraquark states without doubt
  - ▶ $Z_c(3900)$, $Z_c(4020)$, $Z_c(4430)$…

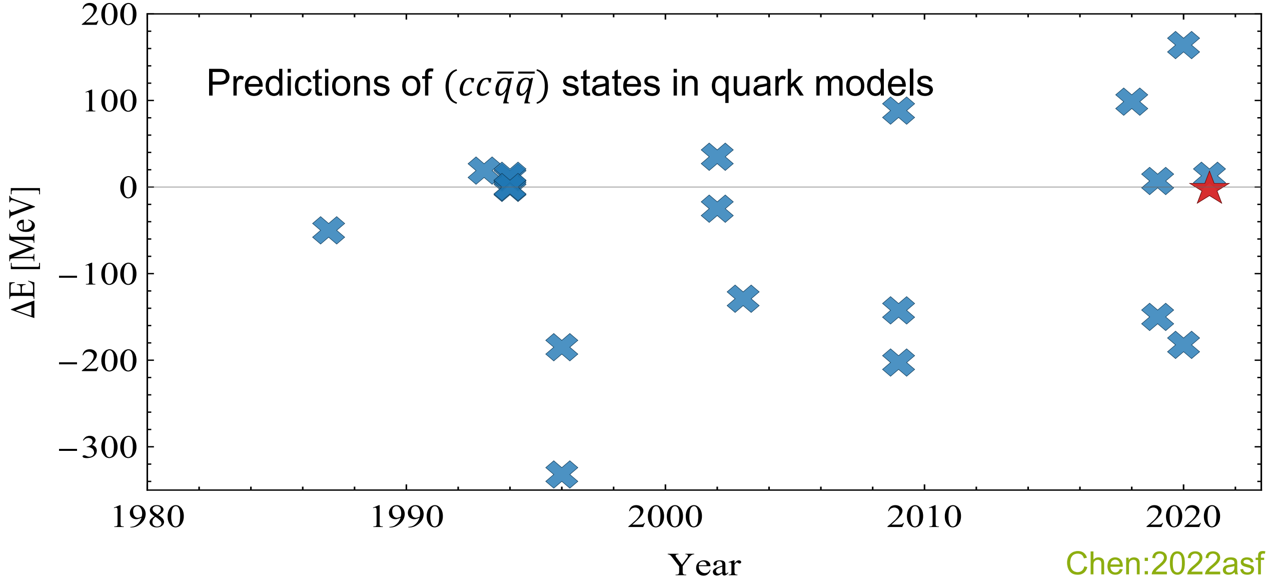
Tetraquark states

- Recently, more and more hadrons composed of at least four quarks were observed

New naming scheme: Gershon:2022xnn

$[c\bar{c}qqq]$ P_c	$[cc\bar{c}\,\bar{c}]$ $X(6900)$	$[cs\bar{u}\,\bar{d}]$ $T_{cs1}(2900)$ $T_{cs0}(2900)$	$[cs\bar{u}\,\bar{d}]$ $Z_{cs}(3985)$ $Z_{cs}(4000)$	$[cc\bar{u}\,\bar{d}]$ $T_{cc}(3875)^+$	$[c\bar{s}u\bar{d}][c\bar{s}\bar{u}d]$ $T_{c\bar{s}0}(2900)^{++}$ $T_{c\bar{s}0}(2900)^0$
	2006.16957 2304.08962 …	2009.00025 2009.00026	2011.07855 2103.01803	2109.01038 2109.01056	2212.02716 2212.02717

- Different quark models predicted different results
  - ▶ Example: T_{cc} states



Chen:2022asf

- What is responsible for variations?

Interaction**S** + few-body method**S**

- Benchmark calculations

Benchmark test calculation of a four-nucleon bound states

Benchmark Test Calculation of a Four-Nucleon Bound State

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Kamada:2001tv

Method	E_b
Faddeev-Yakubovsky Eq.	-25.94(5)
Gaussian basis expansion	-25.90
stochastic variational method	-25.92
Hyperspherical variational	-25.90(1)
Green's function MC/Diffusion MC	-25.93(2)
No-core shell model	-25.80(20)
Hyperspherical harmonic methods	-25.944(10)

Benchmark calculations

- Quark interactions
  - AL1,AP1,SLM
- Few-body methods
  - Gaussian expansion method
  - Resonating group method
  - Diffusion Monte Carlo method
- Over 150 tetraquark systems
  - Fully heavy tetraquark states ($QQ\bar{Q}\bar{Q}$)
  - Triply heavy tetraquark states ($QQ\bar{Q}\bar{q}$)
  - Doubly heavy tetraquarks states ($QQ\bar{q}\bar{q}$)
  - Single heavy strange states ($Qs\bar{q}\bar{q}$, $Q\bar{s}q\bar{q}$)

  $q = u, d, s;\ \ Q = b, c$
- Bound states
- Manifestly exotic
  - Do not involve a mixture with the conventional mesons through the creation and annihilation of $\bar{n}n$, where $n = u, d$.

► $J^P = 0^+, 1^+, 2^+$, Only S-wave

Quark interactions

Quark potential models

- Semay-Silvestre-Brac Models

Semay:1994ht, Silvestre-Brac:1996myf

$$V_{ij}(r) = \left[-\frac{\kappa}{r} + \lambda r^p - \Lambda + \frac{2\pi}{3m_i m_j}\kappa' \frac{1}{\pi^{3/2} r_0^3} e^{(-r^2/r_0^2)} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \right] \lambda_i \cdot \lambda_j$$

AL1: $p = 1$ and AP1: $p = 2/3$

"Predictions"!

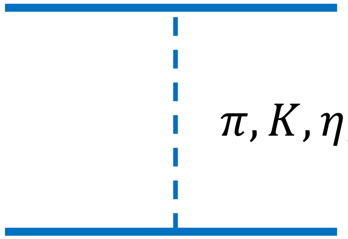
- Chiral quark models [e.g Salamanca model (SLM)]

Vijande:2004he, Gonzalez:2012gka

$$V_{ij}(r) = \left[\frac{\alpha_s}{4} \left(\frac{1}{r} - \frac{1}{6m_i m_j} \frac{e^{-r/r_0}}{r_0^2 r} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \right) + \left(-a_c(1 - e^{-\mu_c r}) + \Delta \right) \right] \lambda_i \cdot \lambda_j$$

Screened confinement

$$+V_\pi + V_K + V_\eta + V_\sigma$$



- In this work, we use AL1, AP1 and SLM

[GeV]	π	K	D	D_s	B	B_s	B_c	η_c	η_b
Exp.	0.139	0.494	1.870	1.968	5.279	5.367	6.274	2.984	9.399
AL1	0.138	0.491	1.862	1.962	5.293	5.361	6.292	3.005	9.424
AP1	0.139	0.498	1.881	1.955	5.311	5.356	6.269	2.982	9.401
SLM	0.140	0.469	1.896	1.983	5.275	5.348	6.275	2.990	9.451

Other QM, e.g.: W.L.Wang, F.Huang, Z.Y.Zhang and B.S.Zou, Phys. Rev. C 84 (2011), 015203

Quark potential models

- Semay-Silvestre-Brac Models

Semay:1994ht, Silvestre-Brac:1996myf

$$V_{ij}(r) = \left[-\frac{\kappa}{r} + \lambda r^p - \Lambda + \frac{2\pi}{3m_i m_j}\kappa' \frac{1}{\pi^{3/2} r_0^3} e^{(-r^2/r_0^2)} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \right] \lambda_i \cdot \lambda_j$$

AL1: $p = 1$ and AP1: $p = 2/3$

- Chiral quark models [e.g Salamanca model (SLM)]

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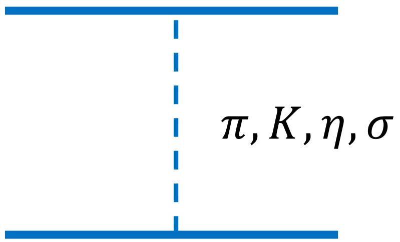
$$V_{ij}(r) = \left[\frac{\alpha_s}{4} \left(\frac{1}{r} - \frac{1}{6m_i m_j} \frac{e^{-r/r_0}}{r_0^2 r} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \right) + \underbrace{\left(-a_c(1 - e^{-\mu_c r}) + \Delta\right)}_{\text{Screened confinement}} \right] \lambda_i \cdot \lambda_j$$

$$+V_\pi + V_K + V_\eta + V_\sigma$$



- In this work, we use AL1, AP1 and SLM

[GeV]	ρ	ω	ϕ	K^*	D^*	D_s^*	B^*	B_s^*	B_c^*	J/ψ	Υ
Exp.	0.775	0.783	1.019	0.892	2.010	2.112	5.325	5.415	6.329	3.097	9.460
AL1	0.770	0.770	1.021	0.903	2.016	2.102	5.350	5.417	6.343	3.101	9.461
AP1	0.770	0.770	1.021	0.908	2.033	2.107	5.367	5.418	6.338	3.102	9.461
SLM	0.773	0.693	1.000	0.902	2.018	2.111	5.317	5.393	6.329	3.097	9.501

Gaussian expansion method

Rayleigh-Ritz Variational method

- Variational method:

$$\mathcal{E}[\psi] = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}, E_{ground} = \mathcal{E}_{min}$$

- The efficiency of variational method depends on the trial functions (basis functions)
- A flexible choice: a set of non-orthogonal basis
- Basis expansion: $|\psi\rangle = a_i|\phi_i\rangle$, $\delta\mathcal{E}[\psi] = 0$ become

$$\begin{aligned}\frac{\partial\mathcal{E}}{\partial a_i} &= 2\sum_j \left[\frac{\langle\phi_i|H|\phi_j\rangle a_j}{\langle\psi|\psi\rangle} - \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle^2} a_j\langle\phi_i|\phi_j\rangle\right] \\ &= 2\sum_j \frac{1}{\langle\psi|\psi\rangle}\left[\langle\phi_i|H|\phi_j\rangle a_j - \mathcal{E} a_j\langle\phi_i|\phi_j\rangle\right] = 0\end{aligned}$$

- Generalized eigenvalue problem

$$\langle\phi_i|H|\phi_j\rangle a_j = \mathcal{E}\langle\phi_i|\phi_j\rangle a_j$$

Gaussian expansion method

- Spatial wave function

$$\phi_{nlm}(\boldsymbol{r}) = N_{lm} r^l e^{-\frac{r^2}{r_n^2}} Y_{lm}(\hat{r}),$$ Geometric progression: $r_n = r_0 a^{n-1}$ Hiyama:2003cu

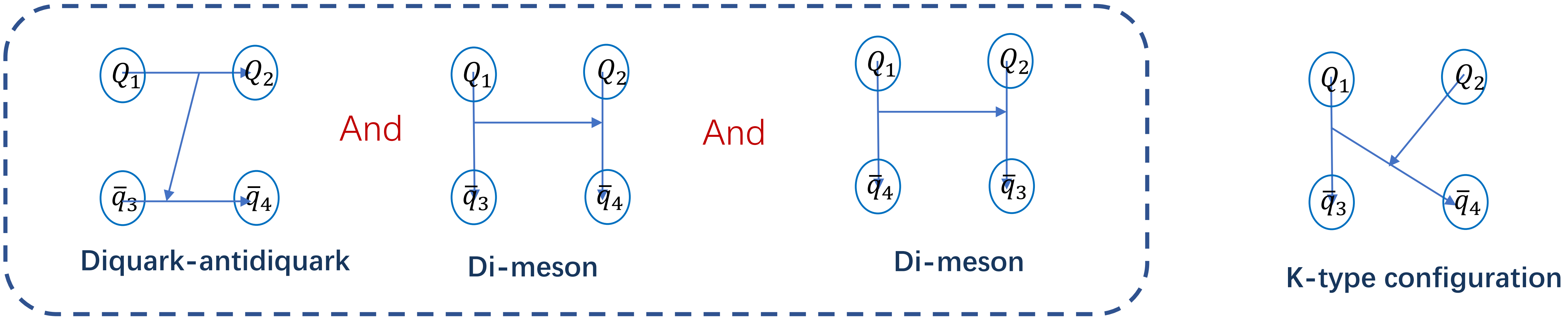
  ▶ non-orthogonal basis but very efficient: embed both long- and short-range correlations

- Jacobi coordinates

  ▶ If we can include the basis completely, only one type of Jacobi coordinates is enough

  Cost: large basis, complicate angular momentum calculations

  ▶ Practical strategy: only S-wave, $l, m = 0$, but choose different type of Jacobi coordinates



Gaussian Expansion Method

- Color functions

$$\begin{cases}[(Q_1Q_2)_{\bar{3}}(\bar{q}_3\bar{q}_4)_3]_1\\ [(Q_1Q_2)_6(\bar{q}_3\bar{q}_4)_{\bar{6}}]_1\end{cases} \quad \text{Or} \quad \begin{cases}[(Q_1\bar{q}_3)_1(Q_2\bar{q}_4)_1]_1\\ [(Q_1\bar{q}_4)_1(Q_2\bar{q}_3)_1]_1\end{cases} \quad \text{Or} \quad \begin{cases}[(Q_1\bar{q}_3)_1(Q_2\bar{q}_4)_1]_1\\ [(Q_1\bar{q}_4)_8(Q_2\bar{q}_3)_8]_1\end{cases}$$

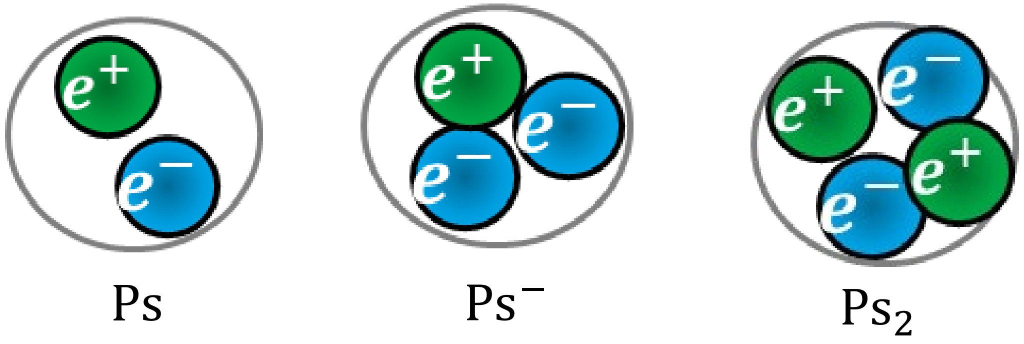
- Spin wave function

$$\begin{aligned}S_{12} &= 0,1;\ S_{34}=0,1\\ S_{12}&\otimes S_{34}\to J\end{aligned} \quad \text{Or} \quad \begin{aligned}S_{13} &= 0,1;\ S_{24}=0,1\\ S_{13}&\otimes S_{24}\to J\end{aligned} \quad \text{Or} \quad \begin{aligned}S_{14} &= 0,1;\ S_{23}=0,1\\ S_{14}&\otimes S_{23}\to J\end{aligned}$$

- Antisymmetrization (e.g. $Q_1 = Q_2$ and $q_3 = q_4$) :

$$\psi = \mathcal{A}\left[\psi_{color}\otimes\psi_{spin}\otimes\psi_{spatial}\otimes\psi_{flavor}\right], \qquad \mathcal{A} = (1-P_{12})(1-P_{34})$$

Di-positronium



- It was predicted to exist in 1946 by J.A Wheeler
- The binding energy is 0.435 485 eV PhysRevLett.92.043401
- It was not observed until 2007 in an experiment 10.1038/nature06094
- Accuracy of our results: 0.435 eV
- However, Assi etal get a result 1.1 eV

Tetraquarks made of sufficiently heavy quarks are bound in QCD

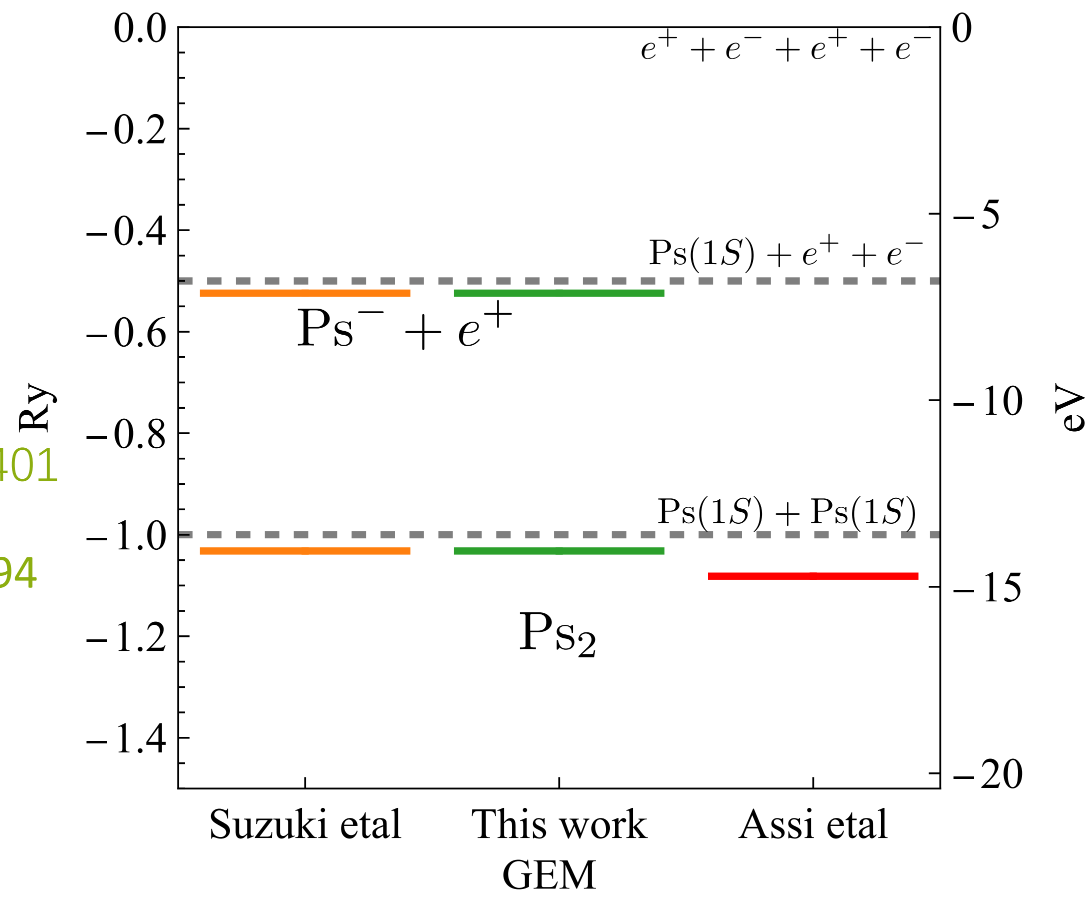
Benoît Assi[1] and Michael L. Wagman[1]

[1] *Fermi National Accelerator Laboratory, Batavia, IL, 60510*

(Dated: November 6, 2023)

Tetraquarks, bound states composed of two quarks and two antiquarks, have been the subject of intense study but have yet to be understood from first principles. Previous studies of fully-heavy tetraquarks in nonrelativistic effective field theories of quantum chromodynamics (QCD) suggest different conclusions for their existence. We apply variational and Green's function Monte Carlo methods to compute tetraquarks' ground- and excited-state energies in potential nonrelativistic QCD. We robustly demonstrate that fully-heavy tetraquarks are bound in QCD for sufficiently heavy quark masses. We also predict the masses of tetraquark bound states comprised of b and c quarks, which are experimentally accessible, and suggest possible resolutions for previous theoretical discrepancies.

arXiv:2311.01498



Di-positronium - Wikipedia

Tetraquark systems

- Fully heavy tetraquark states ($QQ\bar{Q}\bar{Q}$)
- Triply heavy tetraquark states ($QQ\bar{Q}\bar{q}$)
- Doubly heavy tetraquarks states ($QQ\bar{q}\bar{q}$)
- Single heavy strange states ($Qs\bar{q}\bar{q}$, $Q\bar{s}q\bar{q}$)

$q = u, d, s;\ \ Q = b, c$

- $J^P = 0^+, 1^+, 2^+$
  - ▶ Only S-wave

Over 150 states

- In this work, we only focus on bound states

Wang:2019rdo, Meng:2021yjr⋯

	$QQ\bar{Q}\bar{Q}$	$QQ\bar{Q}\bar{q}$	$QQ\bar{q}\bar{q}$	$Qs\bar{q}\bar{q}$	$Q\bar{s}q\bar{q}$
$J^P = 0^+$	No bound	No bound	☺	☺	No bound
$J^P = 1^+$	No bound	No bound	☺	☺	No bound
$J^P = 2^+$	No bound	No bound	☺	☺	No bound

- Masses are shifted to align the theoretical thresholds with the physical ones.

Fully heavy tetraquark states

- The LO pNRQCD: Coulomb interaction
- Assi etal got the bound solutions even from LO
  - ▶ Even for the $cc\bar{c}\bar{c}$
- Our results using the same interaction
  - ▶ Consistent results for DMC and GEM
  - ▶ No bound solutions to $m_Q = 100$ GeV

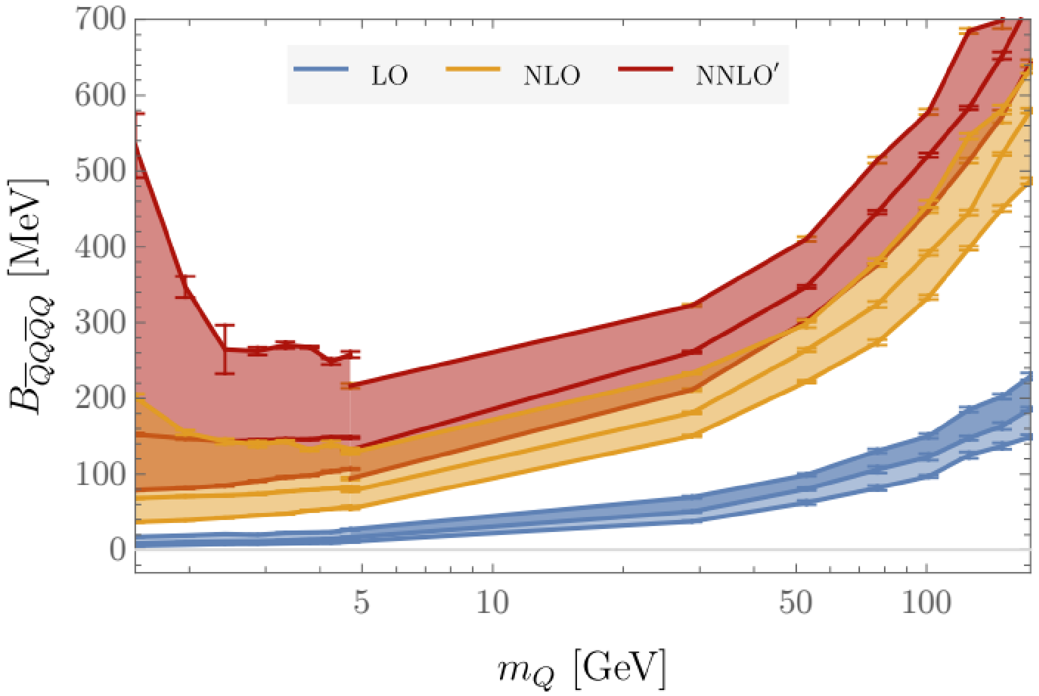


FIG. 2. Tetraquark binding energies for equal-mass quarks with masses ranging from m_c to m_t. Shaded bands connect results with renormalization scale choices $\mu \in \{\mu_p, 2\mu_p, \mu_p/2\}$ at each pNRQCD order indicated.

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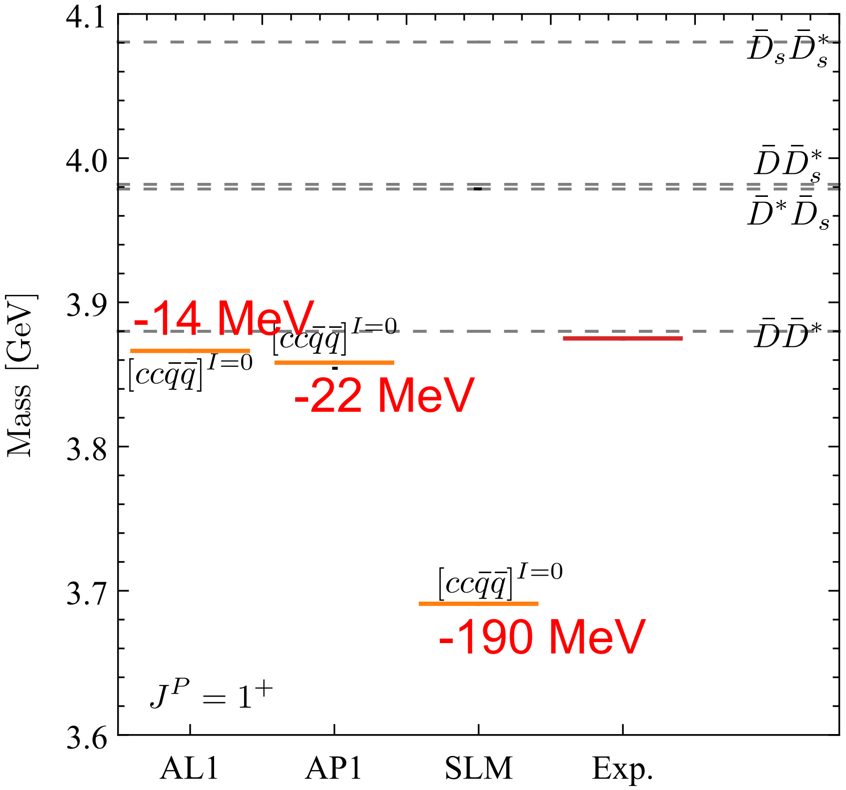
arXiv:2311.01498

$cc\bar{q}\ \bar{q}$ with $J^P = 1^+$

- Points of agreement
  - ▶ $[cc\bar{q}\ \bar{q}]^{I=0}$ bound states ;
- SLM

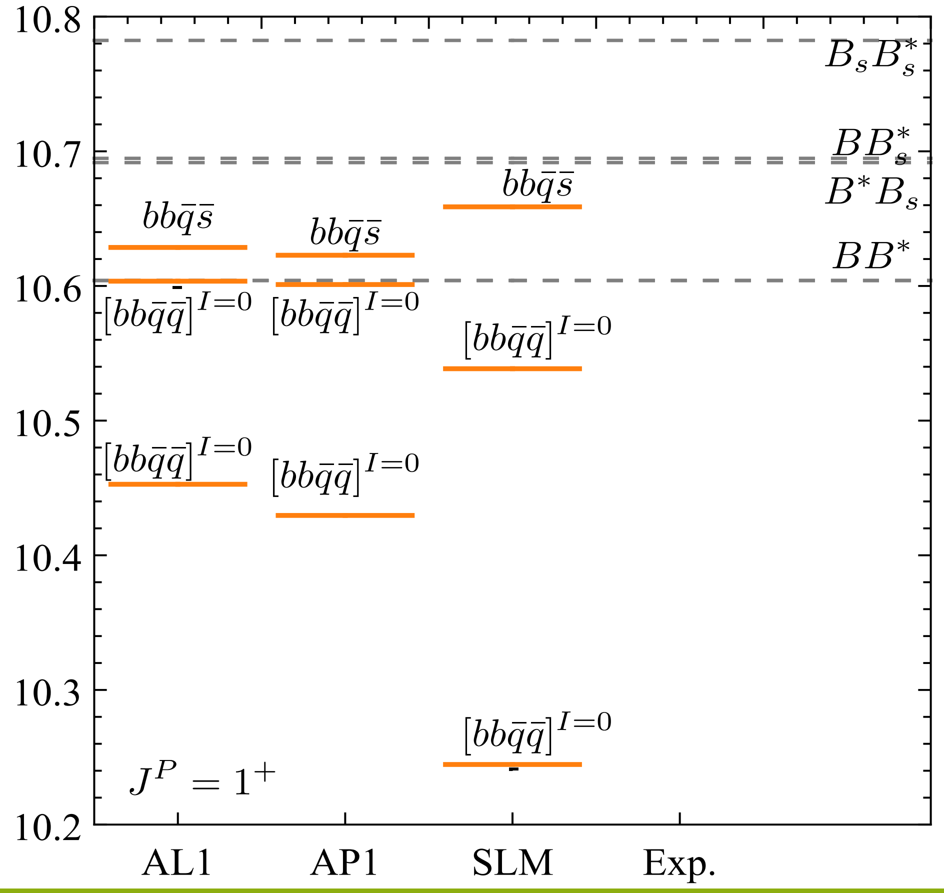
  (1) $[cc\bar{q}\bar{q}]^{I=0}$ are too deep compared with ex. (200MeV VS 200 keV);



"Predictions"

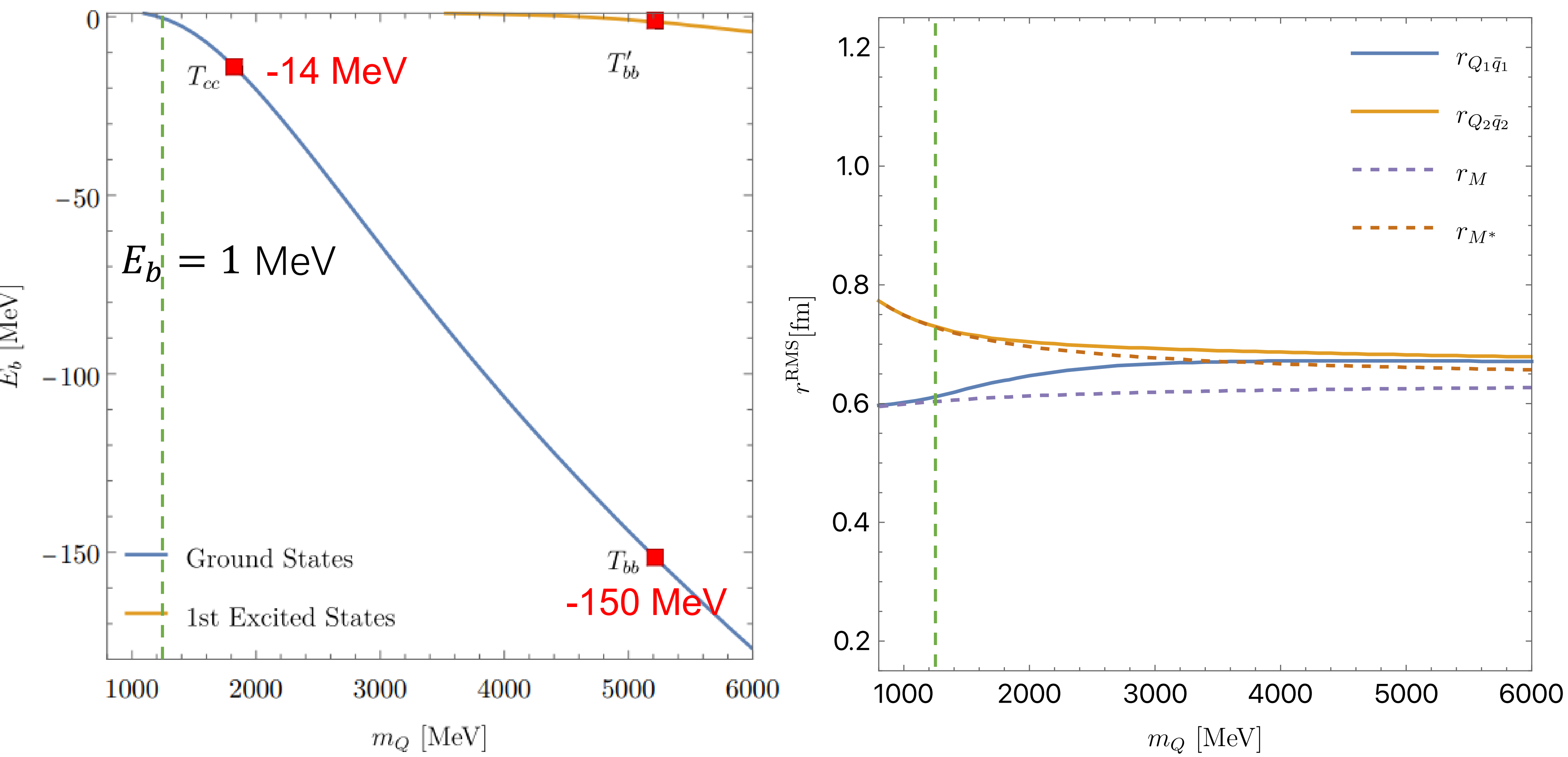
$bb\bar{q}\,\bar{q}$ with $J^P = 1^+$

- Points of agreement
  - ▶ $[bb\bar{q}\,\bar{q}]^{I=0}$ bound states ; *Lattice results: 100-200 MeV*
  - ▶ For $[bb\bar{q}\,\bar{q}]^{I=0}$ systems, the 1st excited states are bound states
  - ▶ $[bb\bar{q}\bar{s}]$ bound states *Lattice results: consistent* $bb\bar{q}\bar{s}$ bound state
  - ▶ No $[bb\bar{q}\,\bar{q}]^{I=1}$ states *Ref.[Ortega:2022efc] :* $[bb\bar{q}\,\bar{q}]^{I=1}$ *bound state! Same SLM interactions!!!*
- SLM
  - ▶ $[bb\bar{q}\bar{q}]^{I=0}$ much deeper than AL1 and AP1



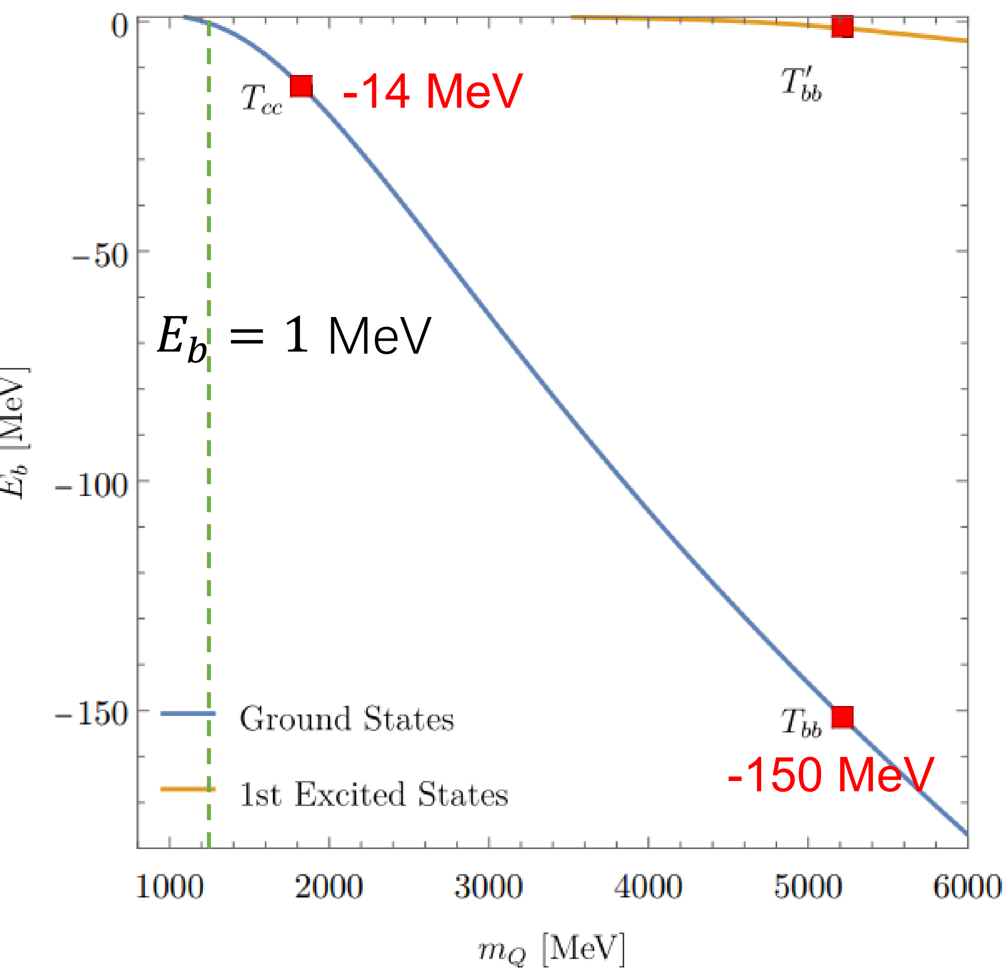
LQCD: Francis:2016hui,Leskovec:2019ioa,Hudspith:2023loy,Junnarkar:2018twb

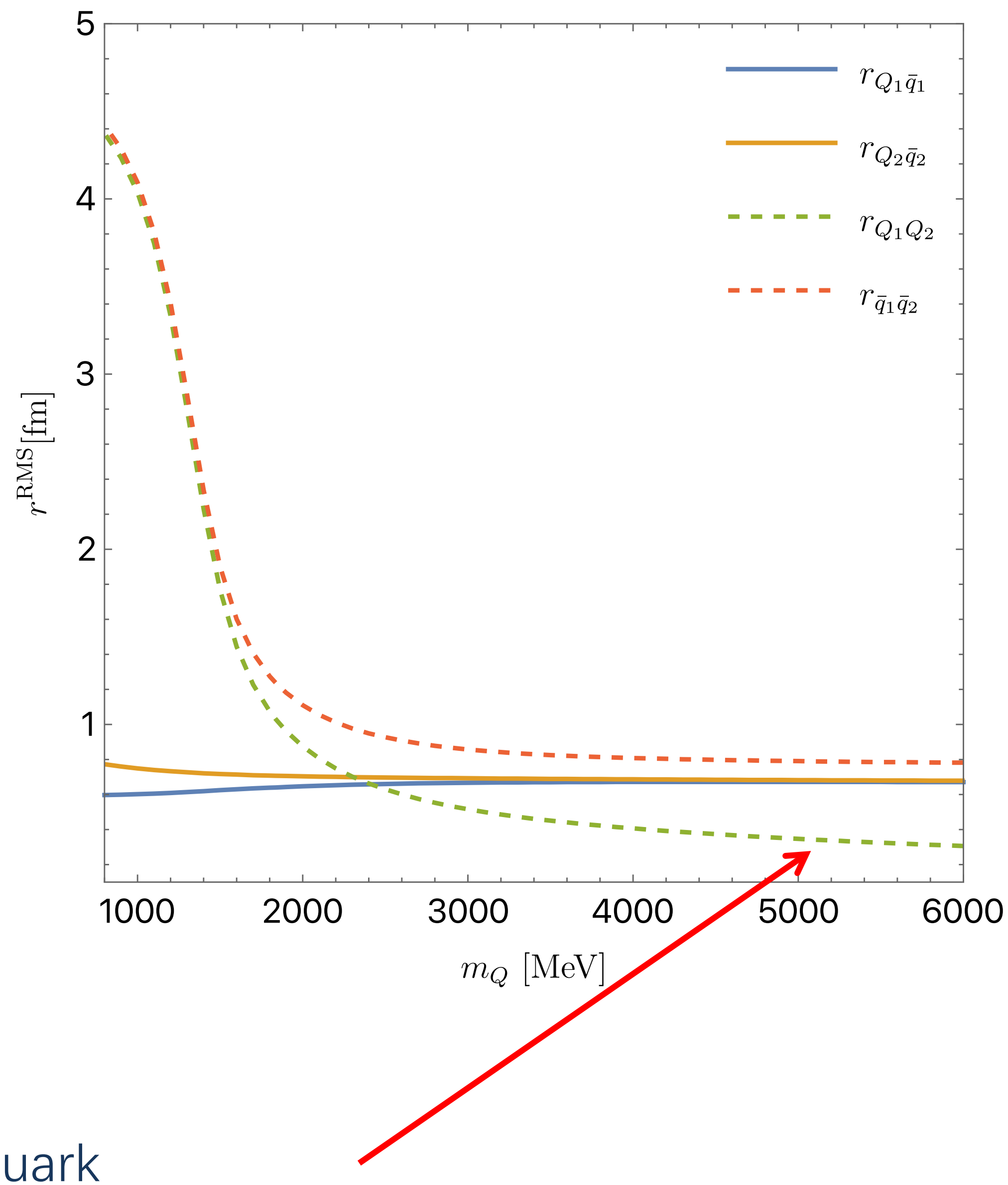
Molecular state: T_{cc}



- Tuning the m_Q to make $E_b < 1$ MeV: molecular states

Diquark in T_{bb}



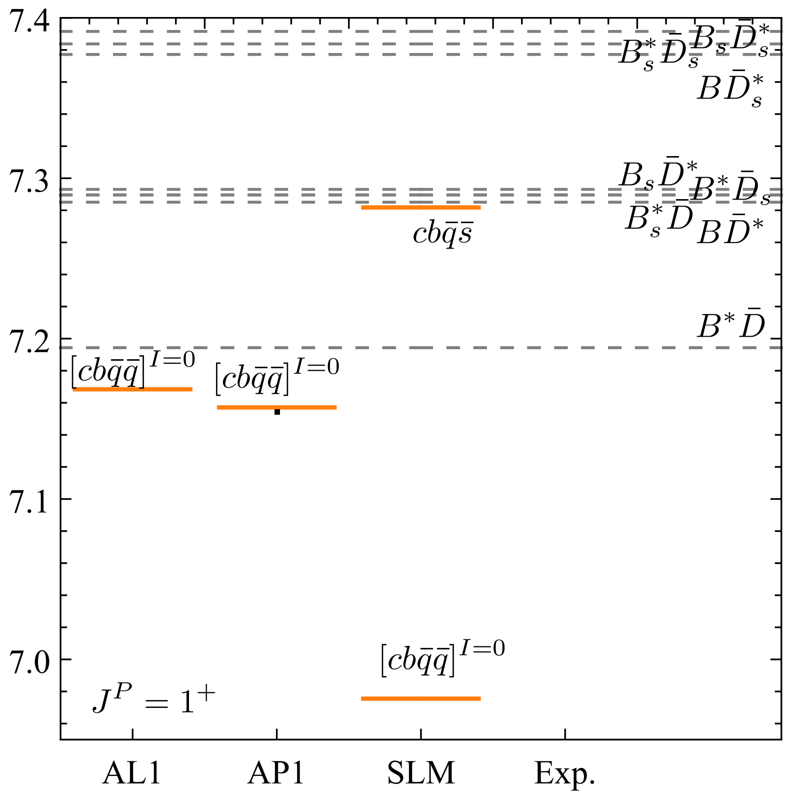


- Tuning the m_Q to m_b: (bb) compact diquark

$bc\bar{q}\,\bar{q}$ with $J^P = 1^+$

- Points of agreement
  - ▶ $[bc\bar{q}\,\bar{q}]^{I=0}$ bound states
- SLM
  - ▶ $[cb\bar{q}\bar{s}]$ bound states

Lattice results: inconclusive



LQCD: Francis:2018jyb,Meinel:2022lzo,Hudspith:2020tdf, Padmanath:2023rdu

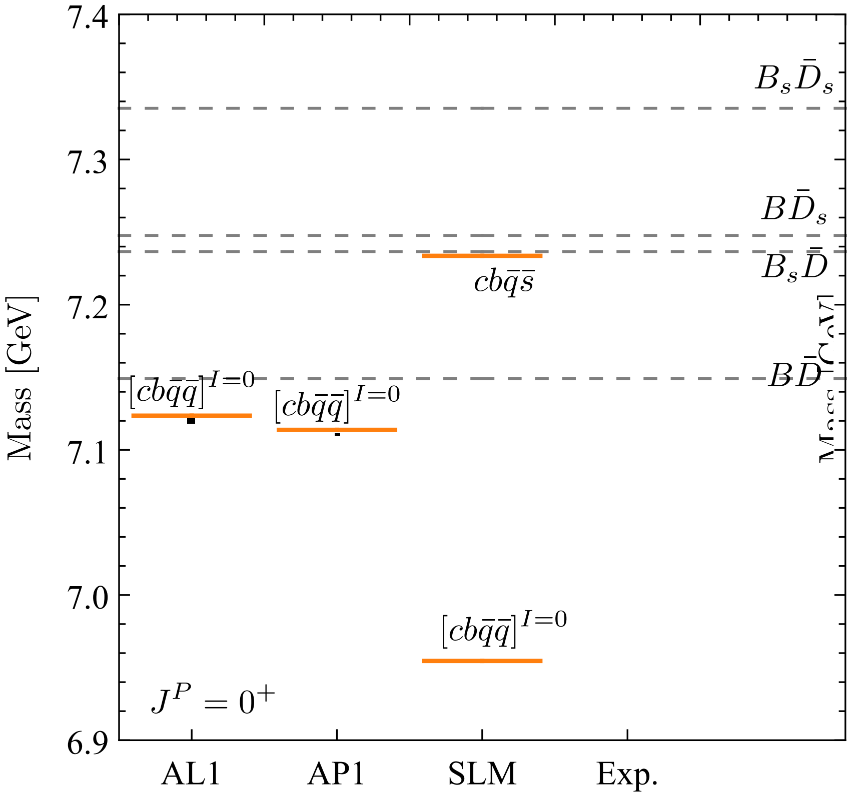
$QQ\bar{q}\,\bar{q}$ with $J^P = 0^+, 2^+$

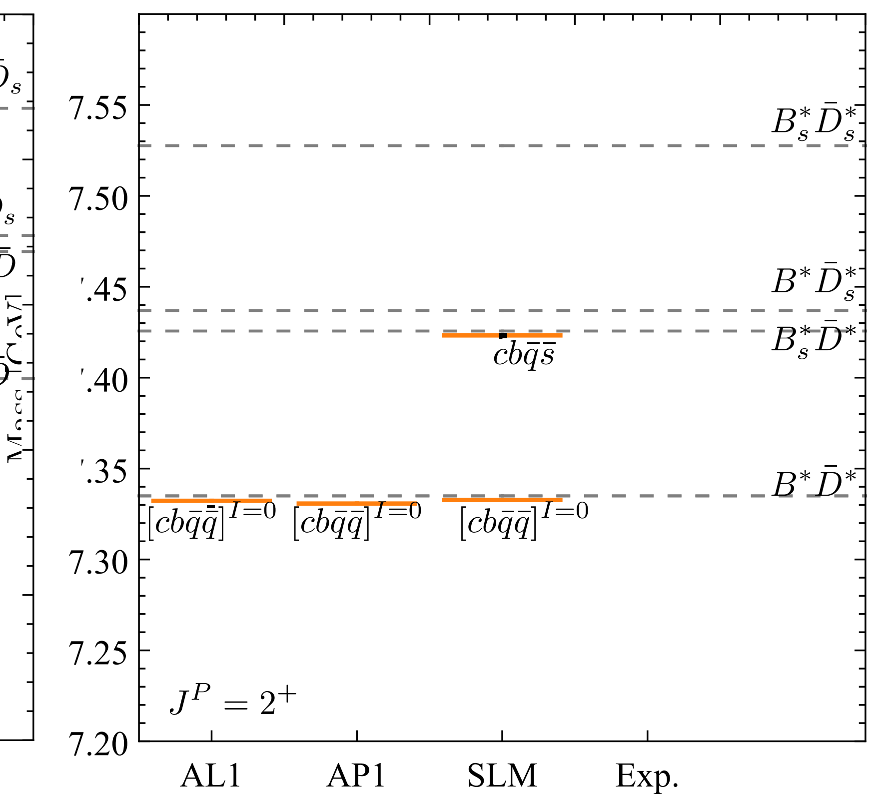
- Points of agreement
  - $[cb\bar{q}\,\bar{q}]^{I=0}$ bound states for $J^P = 0^+, 2^+$
- SLM: $cb\bar{q}\bar{s}$ bound states for $J^P = 0^+, 2^+$
- AP1: $bb\bar{q}\bar{s}$ bound states for $J^P = 2^+$

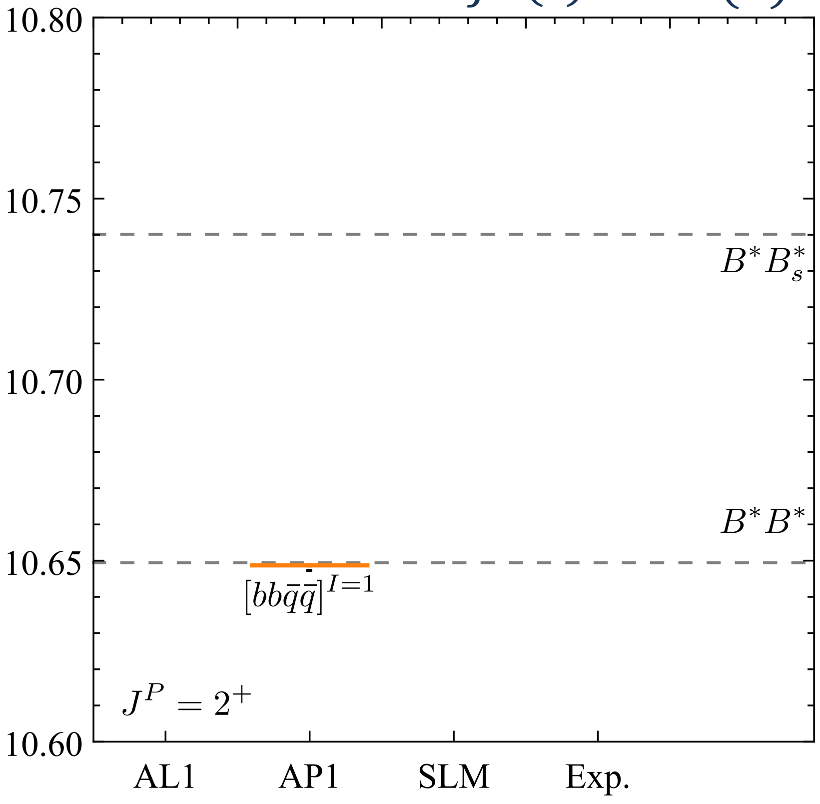
TABLE VII. Properties of the T_{bb} candidates as $B^{(*)}B^{(*)}$ molecules in the $J^P = 0^+$ and 2^+ sectors obtained in this work. Masses, widths, binding energies and partial widths are shown in MeV/c^2. Ortega:2022efc

J^P	I	Mass	Width	E_B	$\mathcal{P}_{BB}$	$\mathcal{P}_{B^*B^*}$	Γ_{BB}	$\Gamma_{B^*B^*}$
0^+	0	10553.0	0	6.0	92%	8%	0	0
		10640.7	2.8	8.7	76%	24%	2.8	0
	1	10545.9	0	13.1	93%	7%	0	0
		10672.6	72.0	−23.2	39%	61%	30.7	41.3
2^+	1	10642.3	0	7.1	-	100%	-	0

The S-wave BB states can not be $J^P(I) = 0^+(0)$

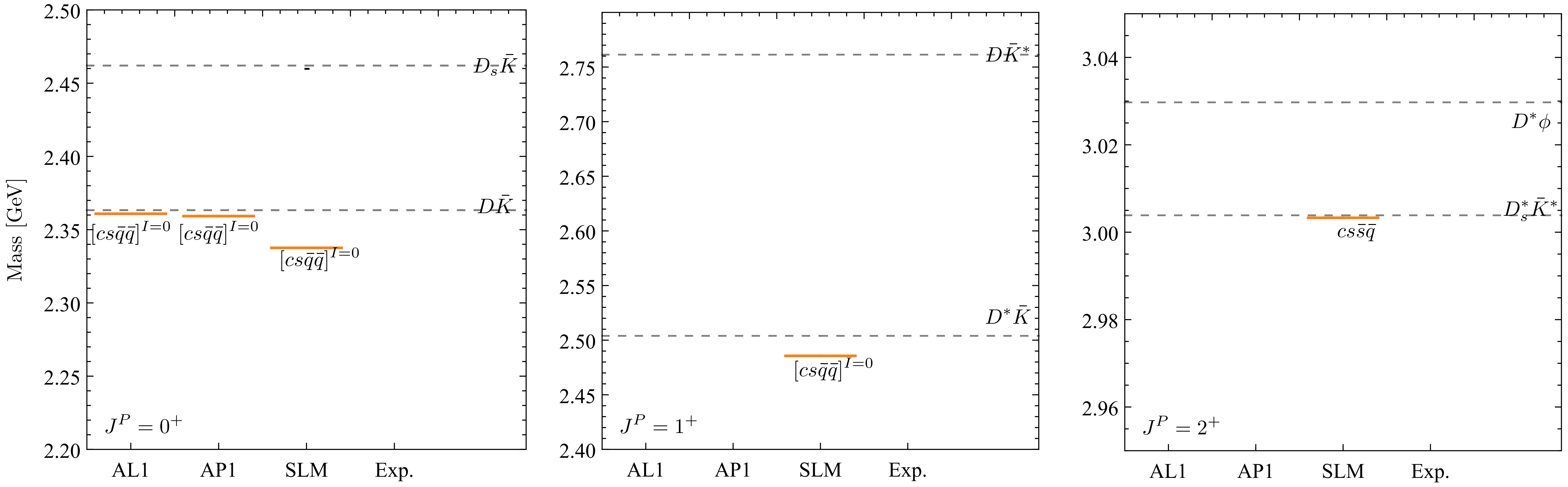






$cs\bar{q}\bar{q}$ systems with $J^P = 0^+, 1^+, 2^+$

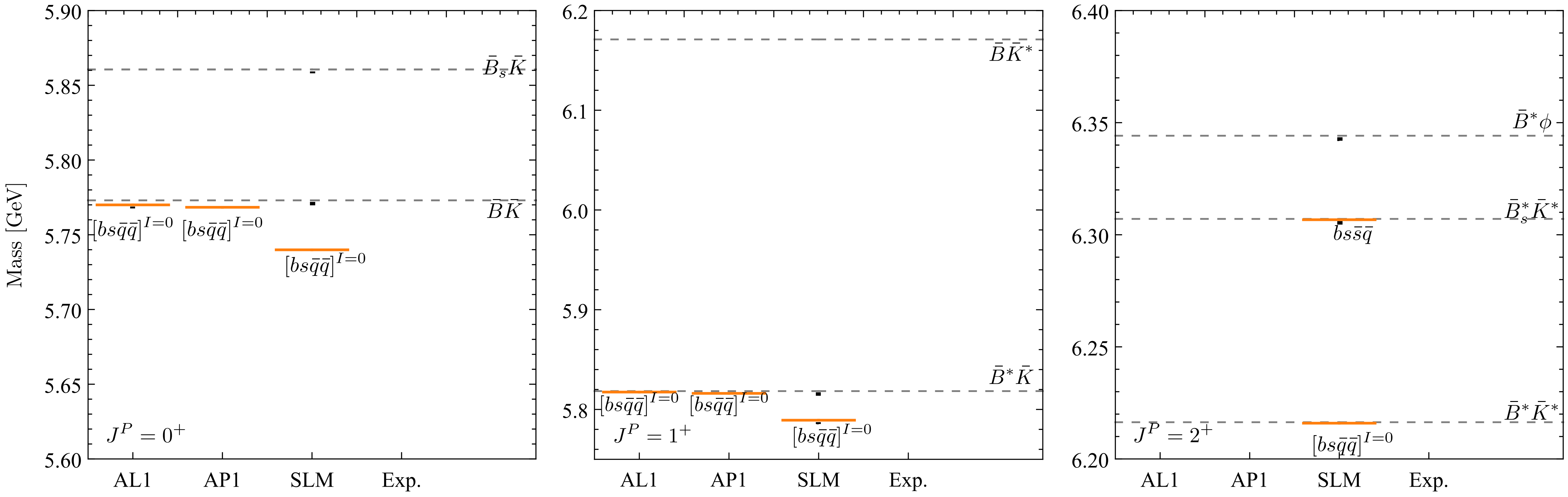
- Points of agreement
  - ▶ $[cs\bar{q}\,\bar{q}]^{I=0}$ bound states for $J^P = 0^+$ *Ortega:2023azl: No $cs\bar{q}\bar{q}$ bound state! Same interactions!!!*
- SLM:
  - ▶ $[cs\bar{q}\bar{q}]^{I=0}$ for $J^P = 1^+$ and $cs\bar{s}\bar{q}$ for $J^P = 2^+$ bound states
- Note: The experimental $T_{cs0}(2900)$ and $T_{cs1}(2900)$ are close to $D^*\bar{K}^*$ thresholds, resonances



$bs\bar{q}\bar{q}$ systems with $J^P = 0^+, 1^+, 2^+$

- Points of agreement
  - ▸ $[bs\bar{q}\,\bar{q}]^{I=0}$ bound states for $J^P = 0^+, 1^+$
- SLM:
  - ▸ $[bs\bar{q}\bar{q}]^{I=0}$ and $bs\bar{s}\bar{q}$ for $J^P = 2^+$ bound states

SLM tends to predict extra states



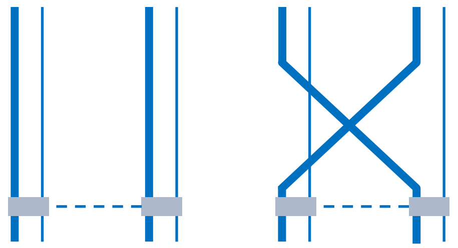
Resonating group method

Resonating Group Method

- Dimeson-wave function

$$\psi_{AB}(\boldsymbol{P}) = \mathcal{A}[\phi_A(\boldsymbol{p}_A)\phi_B(\boldsymbol{p}_B)\chi(\boldsymbol{P})\chi_{AB}^{CST}]$$

  - ▶ ϕ_A and ϕ_B are meson wave functions
  - ▶ We use GEM to get the meson wave functions
  - ▶ $\mathcal{A}$ represents antisymmetriziation operator of identical quarks



- Schrodinger equation of RGM

Entem:2000mq, Ortega:2022efc

$$\left(\frac{\boldsymbol{P}'^2}{2\mu} - E\right)\chi(\boldsymbol{P}') + \int d^3\boldsymbol{P}\left(V_D(\boldsymbol{P}',\boldsymbol{P}) + K_{Ex}(\boldsymbol{P}',\boldsymbol{P})\right)\chi(\boldsymbol{P}) = 0$$

  - ▶ V_D direct interaction, K_{Ex} the exchange kernel

- Compared with GEM
  - ▶ The spin-color-flavor wave functions are complete as well
  - ▶ The RGM neglecting the distortion of the meson wave functions in the tetraquark system
  - ▶ Only the di-meson-type spatial correlations are included
  - ▶ The trial functions are not as general as GEM

$$E_{RGM} \gtrsim E_{GEM}$$

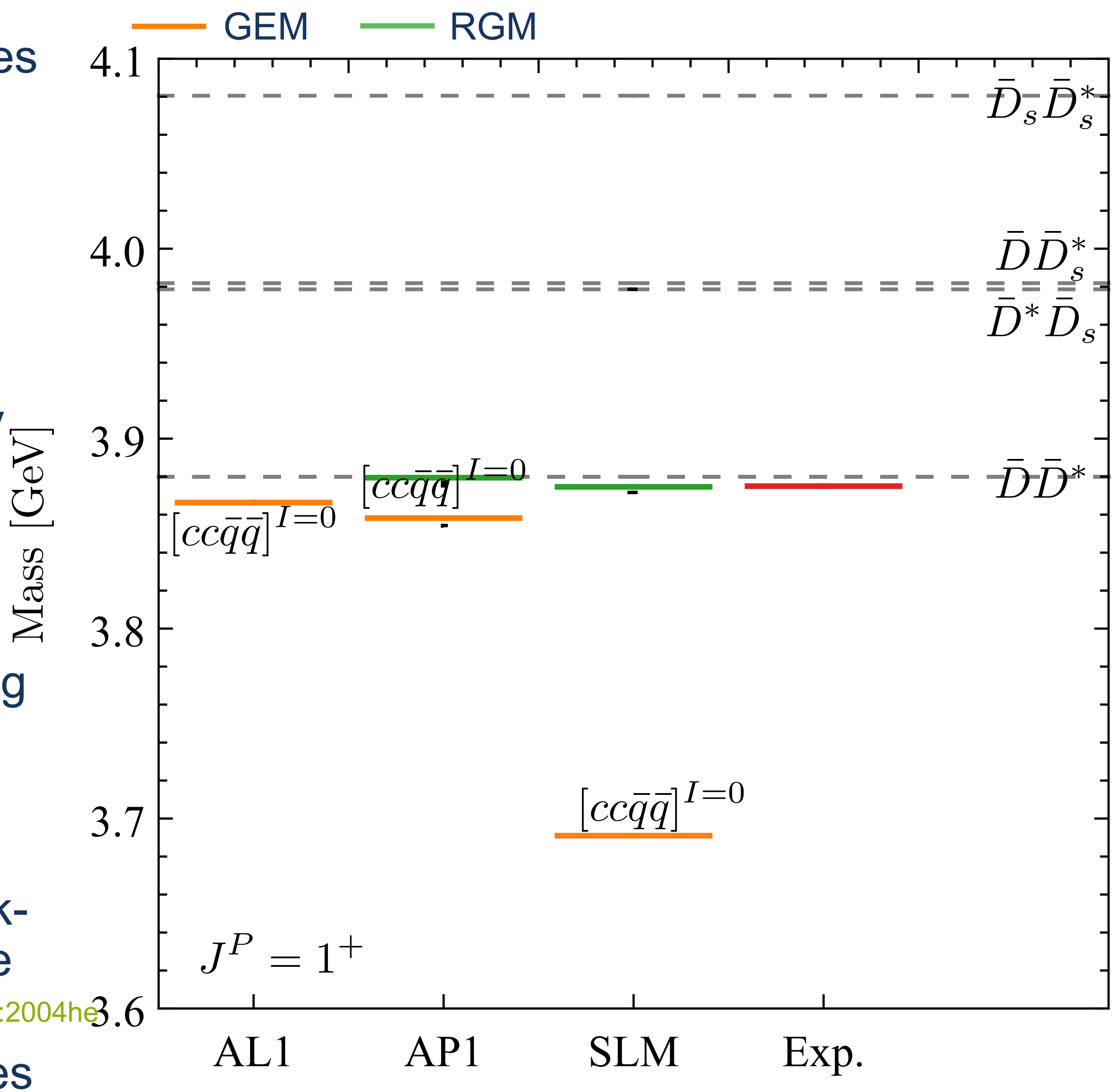
RGM results

- The RGM gives the smaller binding energies
  - ▶ Without the diquark-antidiquark-type correlation
- The RGM results agree with the GEM neglecting diquark-antidiquark correlation
  - ▶ Not general enough trail wave function
  - ▶ Cannot get the ground state accurately
  - ▶ A drawback as a few-body method

However…

- Some quark models (e.g. SLM) constraining the para. using NN phase shifts with RGM
  - ▶ The spatial correlations other than di-hadron types are neglected from birth
  - ▶ Perhaps, it is misleading to use diquark-antidiquark type trial functions for these models
  - ▶ Otherwise, deeper or extra bound states

Entem:2000mq, Vijande:2004he



RGM results

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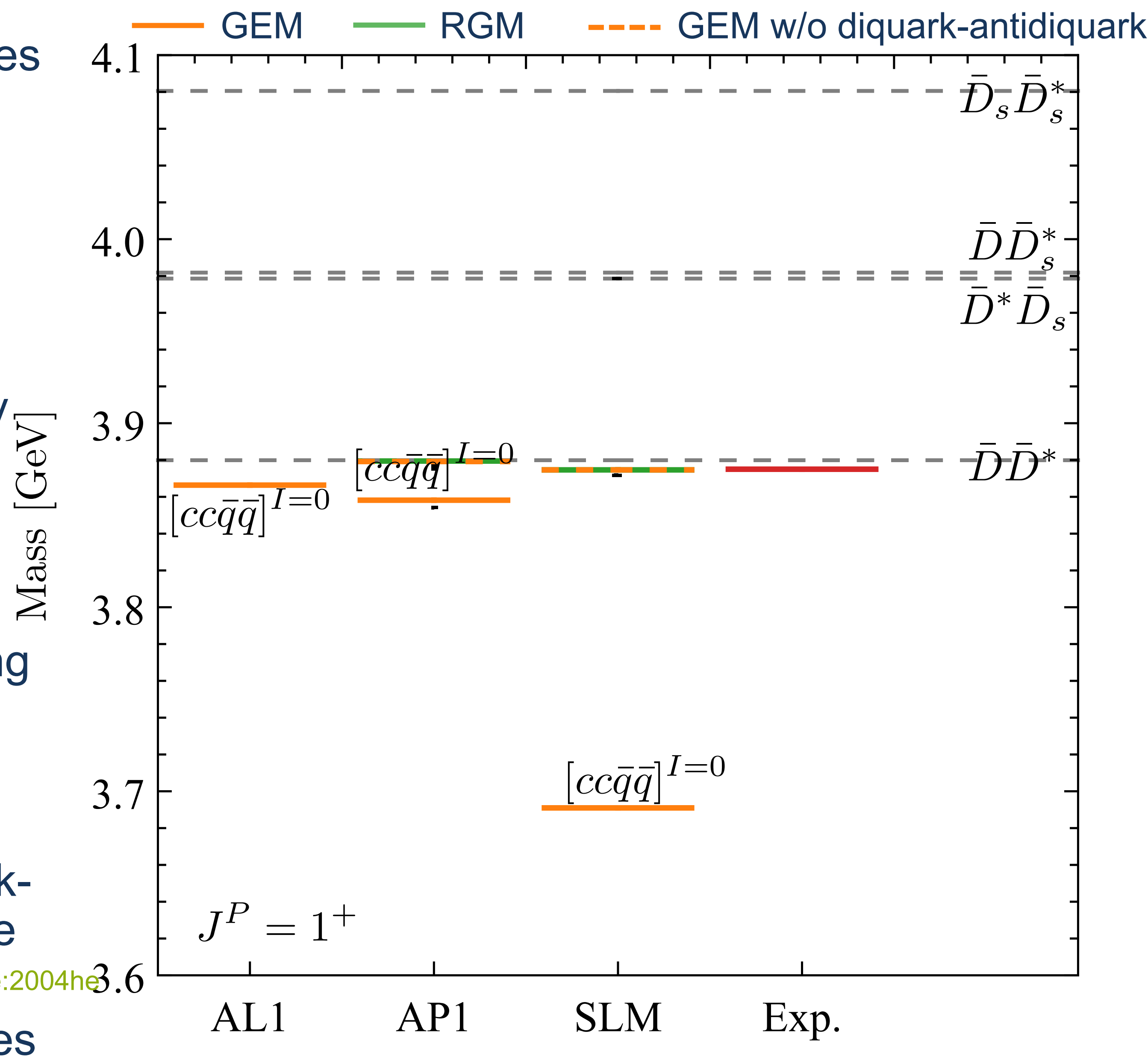


Diffusion monte Carlo method

Diffusion Monte Carlo

- Imaginary Schrödinger equation: $t \to i\tau$

$$-\frac{\partial\Psi(\boldsymbol{R},\tau)}{\partial\tau} = \Big[\underbrace{-\frac{\nabla^2}{2m}}_{\text{Diffusion}} + \underbrace{V(\boldsymbol{R}) - E_R}_{\text{Source or Sink}}\Big]\Psi(\boldsymbol{R},\tau), \quad \Psi(\boldsymbol{R},\tau) = \sum_i c_i\Phi_i(\boldsymbol{R})e^{-[E_i-E_R]\tau},$$

  - Diffusion equation
  - Picture: Salt in a still river
  - If we take $E_R \to E_0$, the $\Psi(\boldsymbol{R},t)$ will approach to the ground state when $t \to \infty$

- The Green's function

$$\psi(\boldsymbol{R},\tau+\Delta\tau) = \int G(\boldsymbol{R},\boldsymbol{R}',\Delta\tau)\psi(\boldsymbol{R}',\tau)d\boldsymbol{R}', \quad G = G_0G_1$$

$$\underbrace{G_0(\boldsymbol{R},\boldsymbol{R}',t) = (2\pi t/m)^{-3/2}e^{-\frac{m(\boldsymbol{R}'-\boldsymbol{R})^2}{2t}}}_{\text{Random Walk}}, \qquad \underbrace{G_1(\boldsymbol{R},\boldsymbol{R}',t) = e^{-\left(\frac{V(R)+V(R')}{2}-E_R\right)t}}_{\text{Death and Birth}}$$

Algorithm

- Walkers: in space D=3N
- Algorithm

1. Sample initial states
2. Random walk: Gaussian distribution

$$(2\pi\Delta\tau/m)^{-3/2}e^{-\frac{m(\Delta \boldsymbol{R})^2}{2\Delta\tau}},$$

3. Death-birth: replicate the walkers n_r times

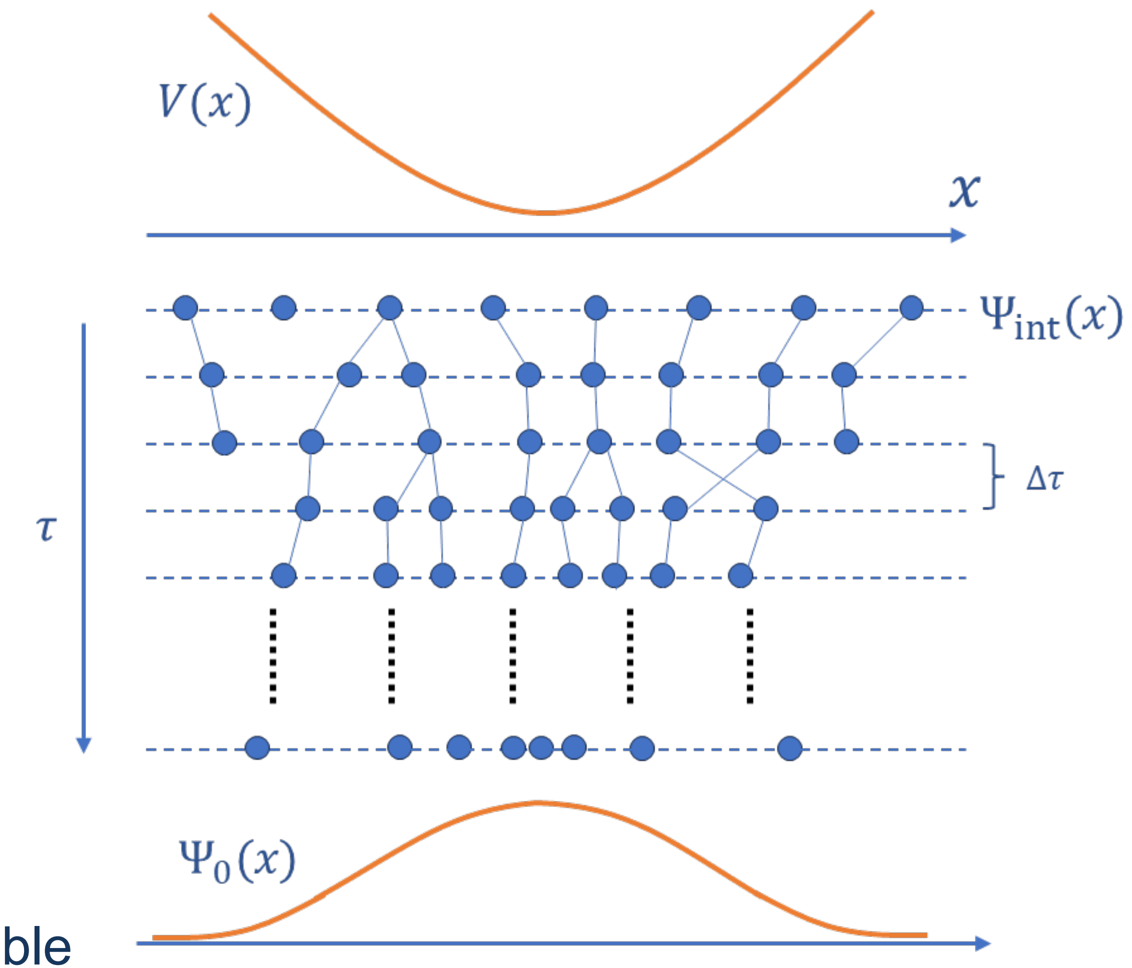
$$n_r = \text{Floor}\left[e^{-\left(\frac{V(R)+V(R')}{2}-E_R\right)\Delta\tau}+u\right]$$

   ► u is uniform distribution in $[0,1]$

4. Repeat 2,3…, until the equilibrium

   ► The distribution and total # of walkers are stable

- No numerical integration
- Manipulates in the 3D Cartesian coordinate, no partial wave expansion, no Jacob coordinate, no complex angles relations



Algorithm

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$$(2\pi\Delta\tau/m)^{-3/2} e^{-\frac{m(\Delta \boldsymbol{R})^2}{2\Delta\tau}},$$

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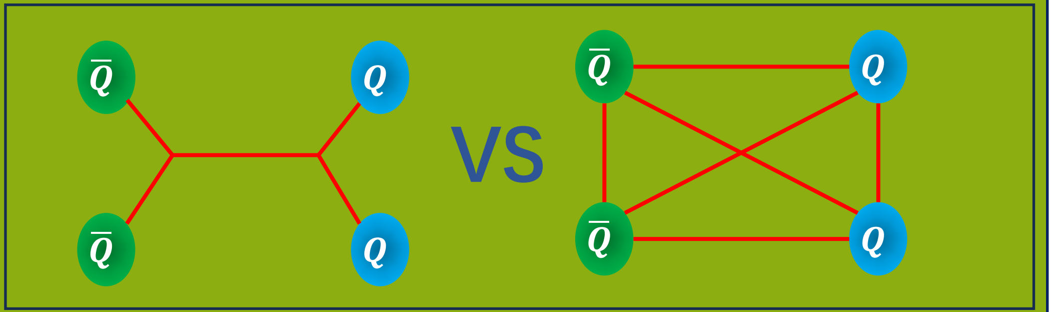
τ

In DMC method, complexity to deal with pairwise confinement interaction and flux-tube interaction are the same!!!



Importance sampling

- Introduce importance function:ψ_T and sample $f(\boldsymbol{R},t)=\Psi(\boldsymbol{R},t)\psi_T(\boldsymbol{R})$
- Schrodinger equation with importance sampling

$$-\frac{\partial f(\boldsymbol{R},t)}{\partial t}=-\underbrace{\sum_{i=1}^{m}\frac{1}{2m_i}\boldsymbol{\nabla}^2_{\boldsymbol{r}_i}f(\boldsymbol{R},t)}_{\text{Random walk}}+\underbrace{\sum_{i=1}^{m}\frac{1}{2m_i}\boldsymbol{\nabla}_{\boldsymbol{r}_i}(\boldsymbol{F}_i(\boldsymbol{R})f(\boldsymbol{R},t))}_{\text{Drift}}+\underbrace{[E_L(\boldsymbol{R})-E_R]f(\boldsymbol{R},t)}_{\text{Sink or source}},$$

  - ▶ $E_L(\boldsymbol{R})=\psi_T(\boldsymbol{R})^{-1}\hat{H}\psi_T(\boldsymbol{R})$ and $\boldsymbol{F}_i(\boldsymbol{R})=2\psi_T(\boldsymbol{R})^{-1}\nabla_{r_i}\psi_T(\boldsymbol{R})=\nabla\ln|\psi_T|^2$
  - ▶ Convection–diffusion equation
  - ▶ Picture: Salt in a flowing river
- The ψ_T should approximate the Ψ_0 as closely as possible
- Green's function of drift term: $G_2(\boldsymbol{R},\boldsymbol{R}',t)=\delta(\boldsymbol{R}-\boldsymbol{R}'-\frac{\boldsymbol{F}(\boldsymbol{R}')}{2m}t)$
  - ▶ make a displacement: $\frac{\boldsymbol{F}(\boldsymbol{R}')}{2m}t$
- Function 1: guiding the walkers to regions of the wavefunction with larger amplitudes.

$$\boldsymbol{F}_i(\boldsymbol{R})=\nabla\ln|\psi_T|^2$$

- Function 2: reduces the fluctuation of the population of walkers

$$E_L(\boldsymbol{R})=\psi_T(\boldsymbol{R})^{-1}\hat{H}\psi_T(\boldsymbol{R})\to E_0$$

Hjorth-Jensen:2017gss, Bai:2016int, Gordillo:2020sgc, Ma:2022vqf

DMC in quark models

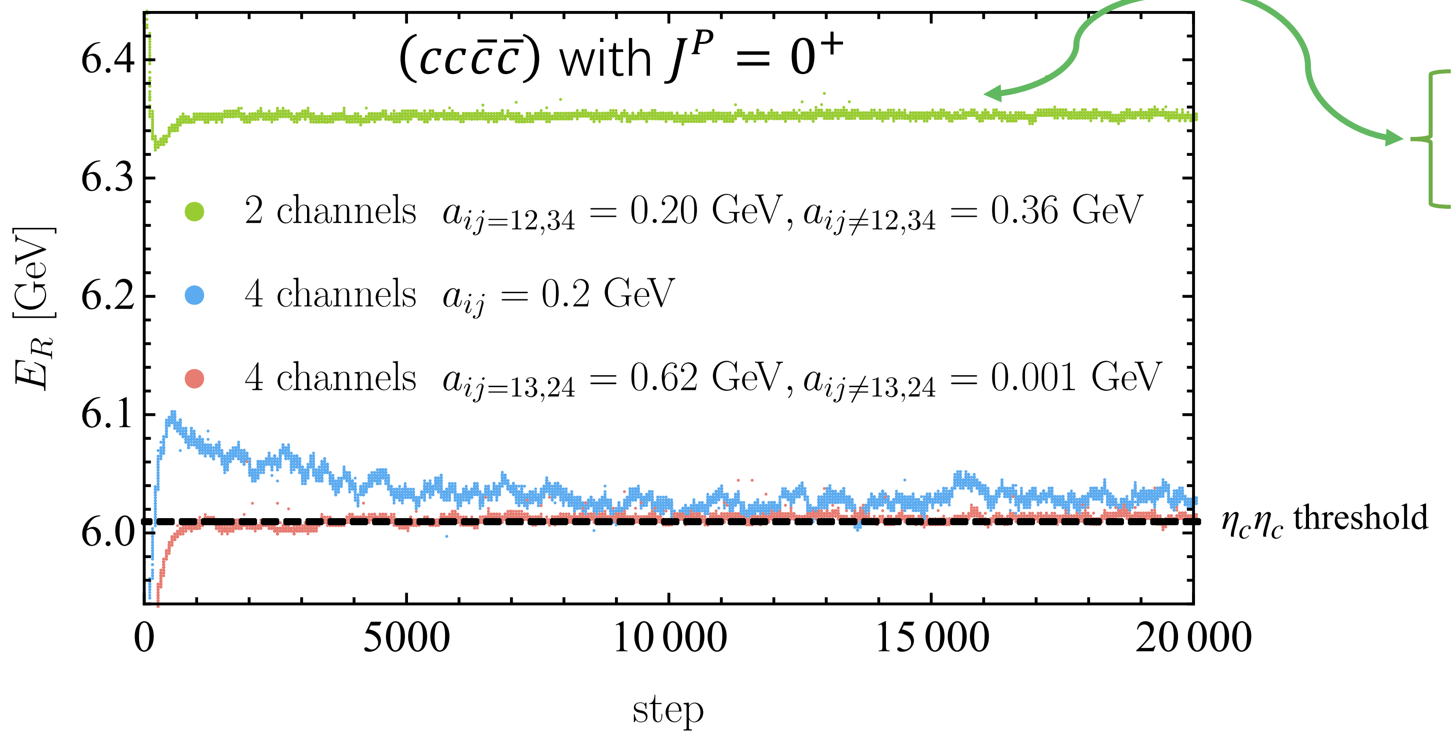
- Cannot get the di-meson thresholds (real ground state) for the systems w/o bound states

Gordillo:2020sgc

- Our advancement: including the extra two channels
  - ▶ Obtain the di-meson thresholds independent of the importance functions

Ma:2022vqf

Gordillo:2020sgc



$$|T1\rangle = \left[(12)^{0_s}_{\bar{3}_c}(34)^{0_s}_{3_c}\right]^{0_s}_{1_c} \psi_1^{SS}(12;34),$$

$$|T2\rangle = \left[(12)^{1_s}_{6_c}(34)^{1_s}_{\bar{6}_c}\right]^{0_s}_{1_c} \psi_2^{SS}(12;34)$$

$$|T3\rangle = \left[(12)^{1_s}_{\bar{3}_c}(34)^{1_s}_{3_c}\right]^{0_s}_{1_c} \psi_3^{AA}(12;34),$$

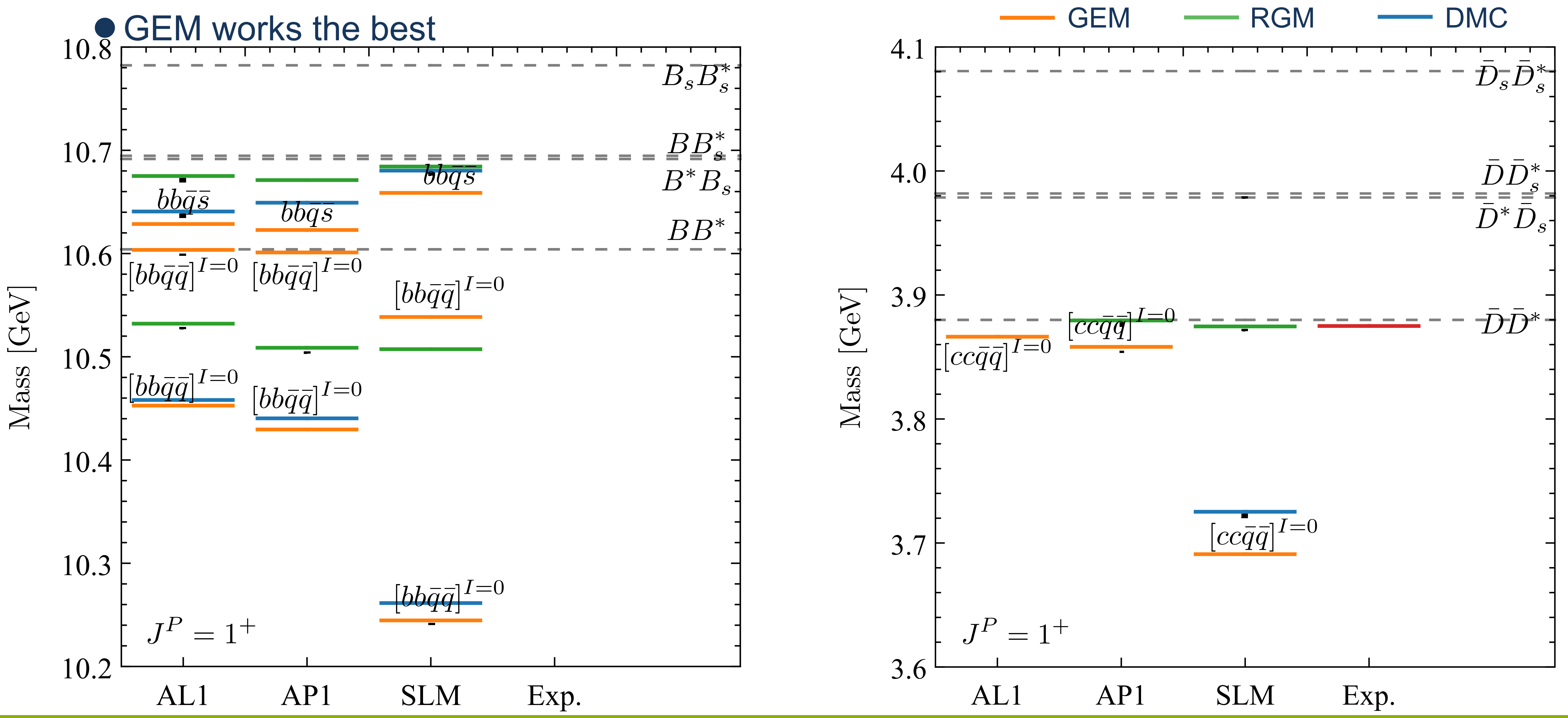
$$|T4\rangle = \left[(12)^{0_s}_{6_c}(34)^{0_s}_{\bar{6}_c}\right]^{0_s}_{1_c} \psi_4^{AA}(12;34)$$

Ours

$$\psi_T(\boldsymbol{R}) = \prod_{i<j} e^{-a_{ij}r_{ij}} .$$

Results from DMC

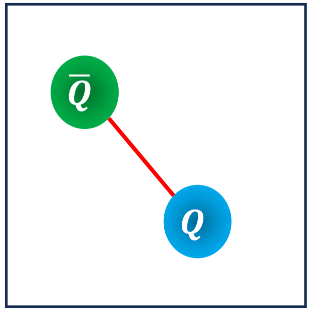
- The DMC with the present coupled-channel strategy give the higher energy than GEM
- The DMC performs better than RGM
- GEM works the best

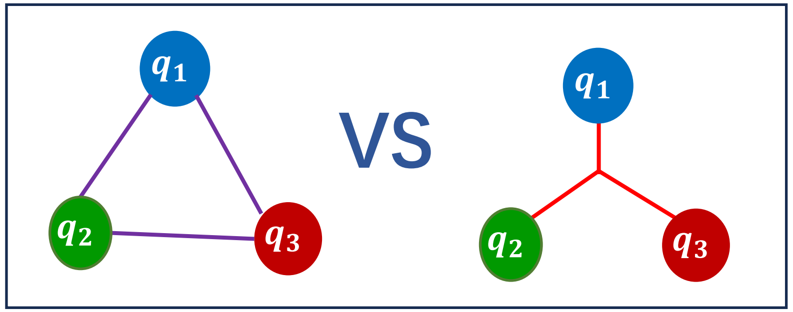


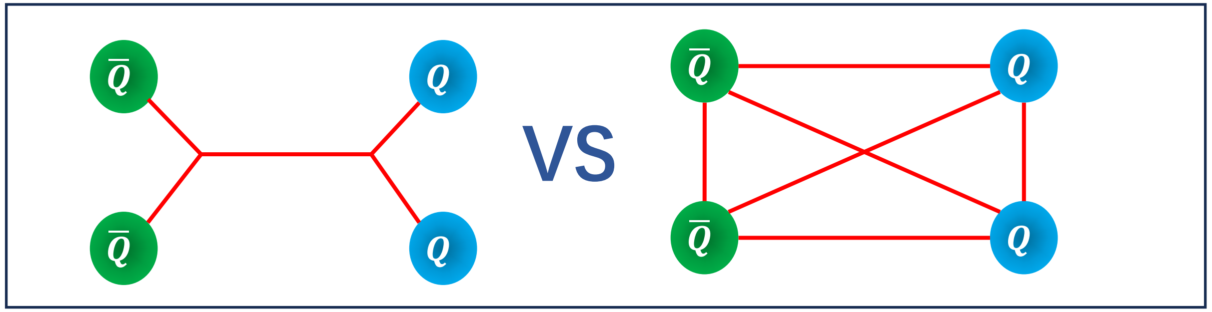
DMC is still a promising method

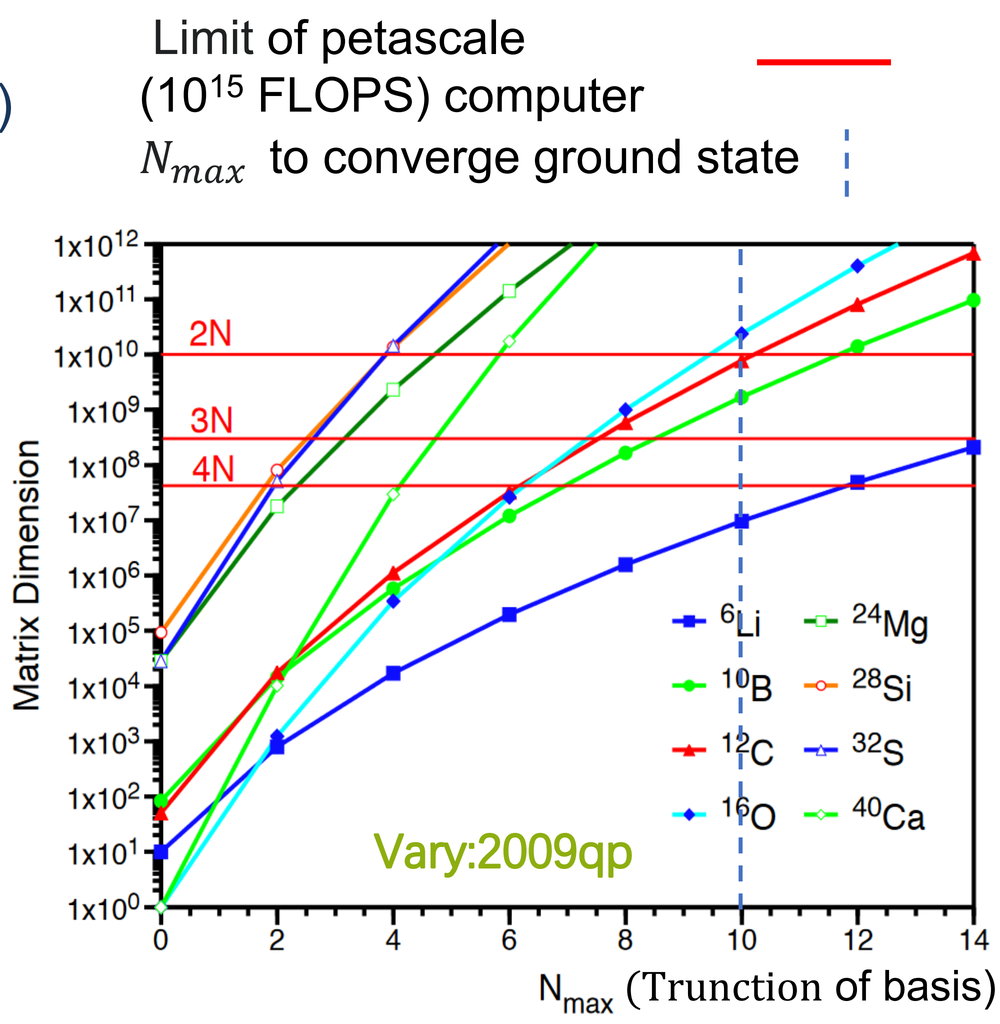
- No presumed clustering
- High precision (in molecular physics and nuclear physics)
- Milder increase computational cost as particles numbers
  - ▶ The computational cost of variational methods increa exponentially with N
- The three-body and four-body force
  - ▶ Flux-tube confinement

Takahashi:2002bw





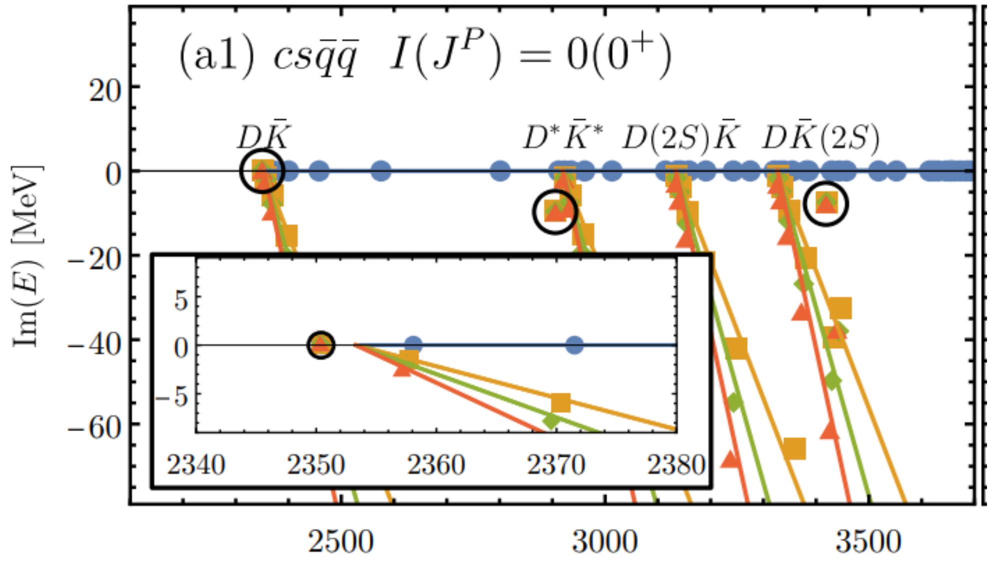




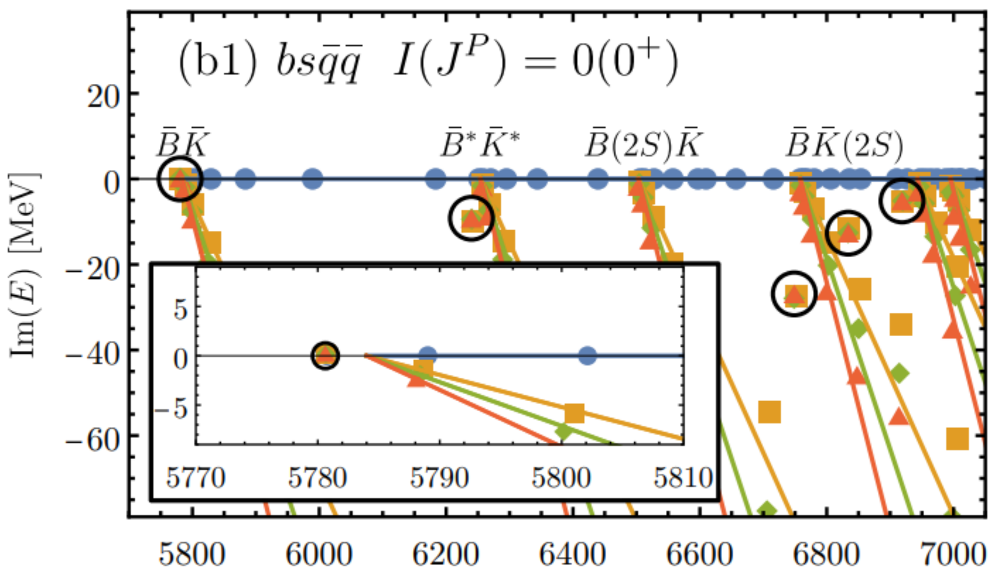
Example of No-core-shell model: A variational method using HO basis)

$Qs\bar{q}\bar{q}$ resonances

- Complex scaling method+GEM: resonance
- There is a $J^P = 0^+$ resonance near the $D^* \bar{K}^*$
- Molecular-type resonance



States	E	ΔM	$r^{\rm RMS}_{c\bar{q}_1}$	$r^{\rm RMS}_{s\bar{q}_2}$	$r^{\rm RMS}_{cs}$	$r^{\rm RMS}_{\bar{q}_1\bar{q}_2}$	$r^{\rm RMS}_{c\bar{q}_2}$	$r^{\rm RMS}_{s\bar{q}_1}$
$D\bar{K}$	2353		0.61	0.59				
$D^*\bar{K}$	2507		0.70	0.59				
$D\bar{K}^*$	2766		0.61	0.81				
$D^*\bar{K}^*$	2920		0.70	0.81				
$0(0^+)$	2350	−3	0.61	0.59	2.45	2.52	2.47	2.50
	$2906-10i$	−14	0.74	0.86	1.12	1.26	1.21	1.27
	$3419-7i$		0.91	1.09	0.87	1.22	1.06	1.09



Summary and outlook

- Investigate the tetraquark bound states with (AL1,AP1,SLM)⊗(GEM,RGM,DMC)
  - ► $(QQ\bar{Q}\bar{Q})$, $(QQ\bar{Q}\bar{q})$, $(QQ\bar{q}\bar{q})$, $(Qs\bar{q}\bar{q})$, $(Q\bar{s}q\bar{q})$

Point out some inconsistencies in literature

- Recommended tetraquark states below di-meson thresholds (consistent predictions of 3 models)

$J^P = 1^+$	$[cc\bar{q}\bar{q}]^{I=0}$	$[bb\bar{q}\bar{q}]^{I=0}$	$[bc\bar{q}\bar{q}]^{I=0}$	$bb\bar{q}\bar{s}$	$[bs\bar{q}\bar{q}]^{I=0}$
$J^P = 0^+$	$[cb\bar{q}\bar{q}]^{I=0}$	$[cs\bar{q}\bar{q}]^{I=0}$	$[bs\bar{q}\bar{q}]^{I=0}$		
$J^P = 2^+$	$[cb\bar{q}\bar{q}]^{I=0}$				

- The trial functions of RGM are not general enough to give the ground state
  - ► For quark models born with RGM, it is inconsistent to include diquark-antidiquark correlations
- DMC: improved to give the di-meson threshold
  - ► By now, has no advantages for tetraquark bound states compared with GEM
- Outlook:
  - ► Pentaquark, hexaquarks states benchmark calculations (sharing the code with the community?)
  - ► Resonances

Albaladejo:2021vln,

  - ► DMC: promising

    Auxiliary field diffusion Monte Carlo

Gandolfi:2007ed

    Flux-tube confinement potentials

Thanks for your attention!!

Backup

Color-electric interaction for multiquark states

- For color-singlet multiquark states $\{Q_1, Q_2, ..., Q_n\}$, $Q_i = Q$ or $\bar{Q}$, if two-body interaction $V_{Q_iQ_j} = V_8(r_{ij})\lambda_i \cdot \lambda_j$, then

$$V_{\{Q_1,Q_2,...Q_n\}} = \sum_{i<j} a_{ij} V_8(r_{ij}), \quad \sum_{i<j} a_{ij} = -\frac{8}{3}n$$

Proof: $2\langle\sum_{i<j} \lambda_i \cdot \lambda_j\rangle = \langle\sum_i \lambda_i\rangle^2 - \sum_i \langle\lambda_i\rangle^2$

- A general problem: For fixed $\sum_{i<j} a_{ij}$, what distribution of a_{ij} give the lowest mass? Walter Thirring, E.M. Harrell, Quantum mathematical physics: atoms, molecules and large systems

$\Rightarrow$ the symmetric case gives the worse result: $M_n(a_{ij}) \leq M_n^{(S)}$

Proof:

$$H = H^S + \Delta V = H^S + aV_8(r_{12}) - aV_8(r_{34}), \quad \langle\psi^S|\Delta V|\psi^S\rangle = 0, \quad (32)$$

$$\langle\psi^S|H^S|\psi^S\rangle = \langle\psi^S|H|\psi^S\rangle \geq \min_{\psi\in\mathcal{H}}\langle\psi|H|\psi\rangle \quad (33)$$

- Intuitively, less symmetric a_{ij}, more deeply bound ground states

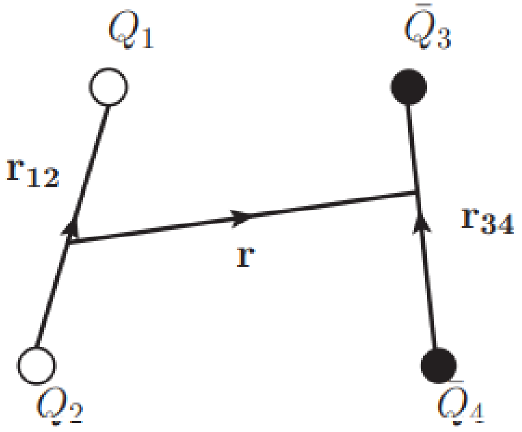
Phys.Rev. D25 (1982) 2370

Color-electric interaction for $QQ\bar{Q}\bar{Q}$

- $\{a_{ij}\}$ distribution for $QQ\bar{Q}\bar{Q}$

a_{ij}	$a_{12}=a_{34}$	$a_{13}=a_{24}$	$a_{14}=a_{24}$
Di-meson	0	$-\frac{16}{3}$	0
$\bar{3}_c-3_c$	$-\frac{8}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$
$\bar{6}_c-6_c$	$\frac{4}{3}$	$-\frac{10}{3}$	$-\frac{10}{3}$

$$2M(Q\bar{Q}) < M(6-\bar{6}) < M(3-\bar{3})$$



- Things become different, when
  - ⇒ unequal quark masses e.g. $QQ\bar{q}\bar{q}$ PRL118 142001; PRL119 202001; PRL119 202002
  - ⇒ hyperfine correction, e.g. color-magnetic interaction $S_i\cdot S_j\lambda_i\cdot\lambda_j$
  - ⇒ multibody interaction, e.g. doubly-Y interaction
  - ⇒...

Phys.Rev. D25 (1982) 2370

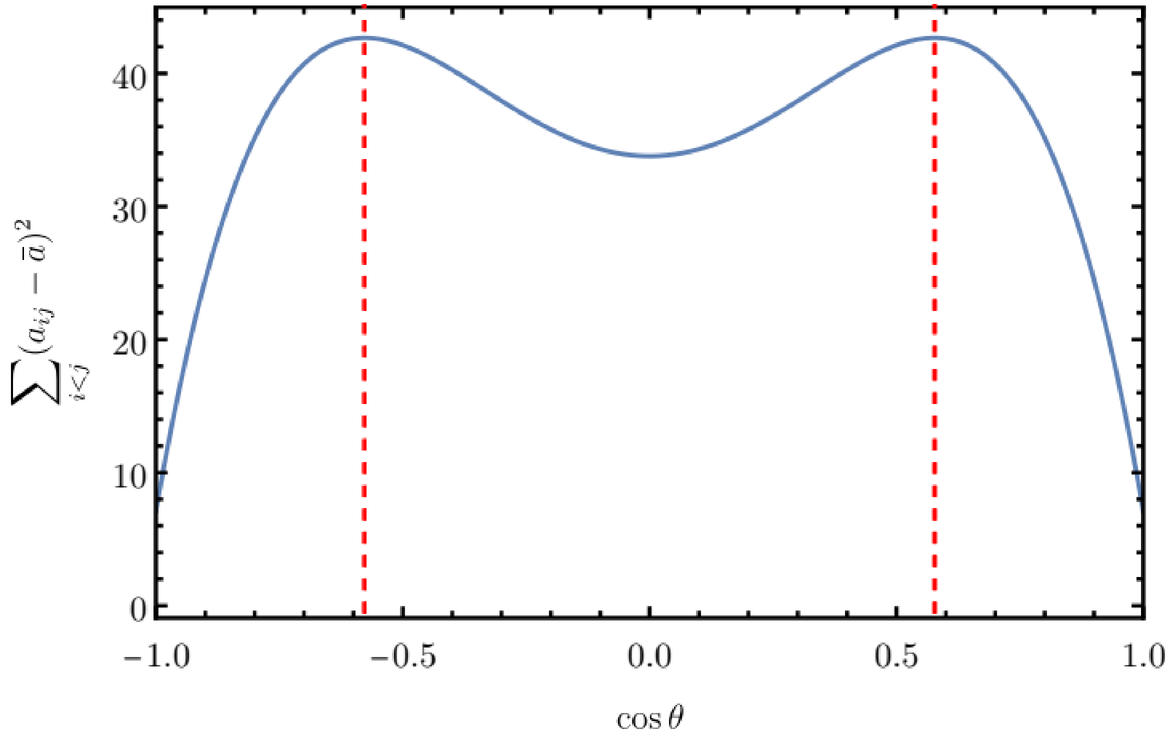
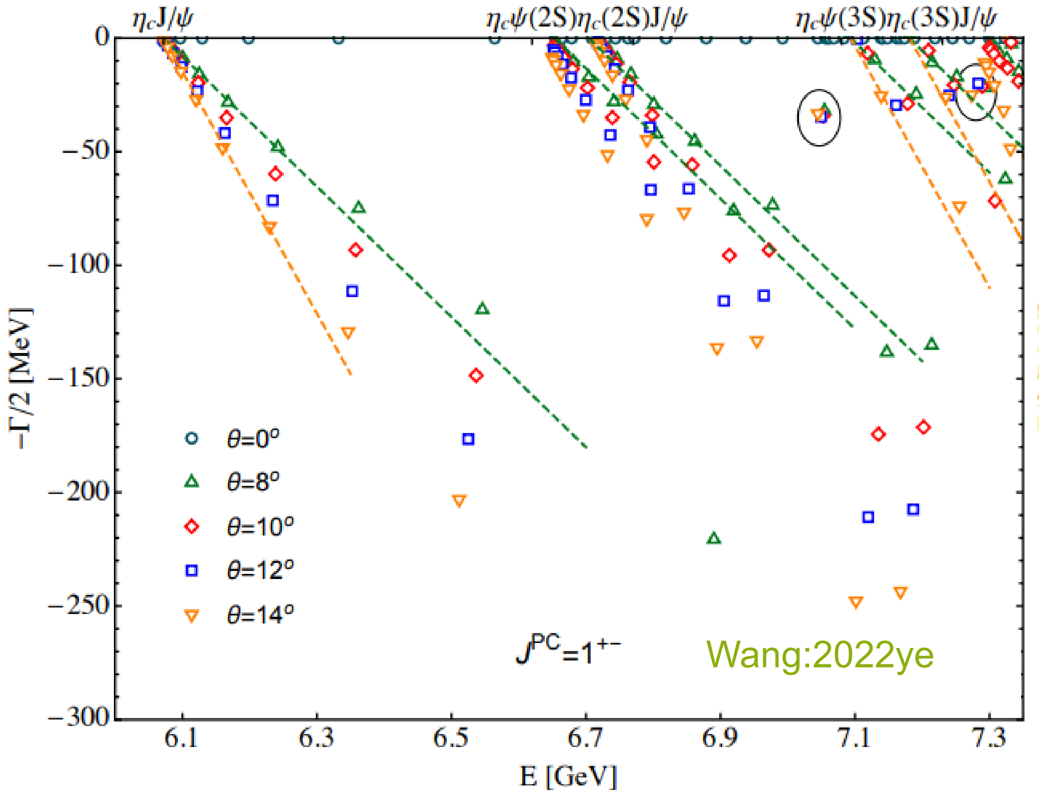


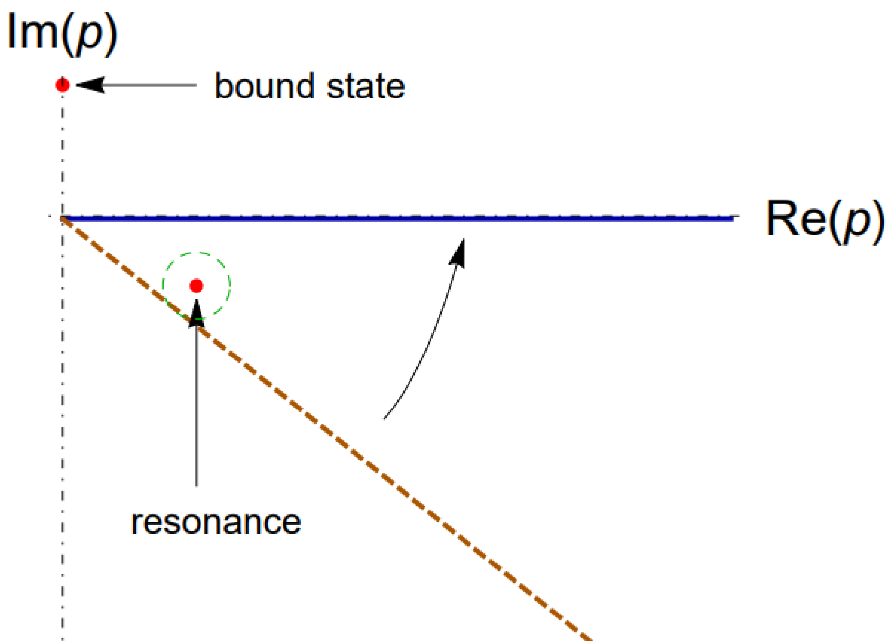
FIG. 1. The variation of $\sum_{i<j}(a_{ij}-\bar{a})^2$ with the mixing angle θ for the tetraquark state described by Eq. (6). The red dashed lines represent $\cos\theta = \pm\frac{1}{\sqrt{3}}$, corresponding to the color wave function $\left|(Q\bar{Q})_{1_c}(Q\bar{Q})_{1_c}\right\rangle$.

Methods to obtain resonance and virtual states

- Complex scaling methods with GEM
  - ▶ It is hard to detect the higher states
  - ▶ The unclear relation with Riemann surface
  - ▶ The tetraquark resonance: two-body scattering problems (confinement)
- RGM + Complex Scaling in coupled-channel two-body problem



Solving Freedholm determinant⇒ Eigenvalue problem



Comparison

TABLE VI. Mass and binding energy (in MeV/c^2) and probabilities of each channel (in %) for the $J^P = 1^+$ T_{bb} states predicted in this work.

Mass	E_B	$\mathcal{P}_{B^0B^{*+}}$	$\mathcal{P}_{B^+B^{*0}}$	$\mathcal{P}_{B^{*+}B^{*0}}$	$\mathcal{P}_{I=0}$	$\mathcal{P}_{I=1}$
10582.2	21.9	47.8	50.0	2.2	99.99	0.01
10593.5	10.5	51.0	48.6	0.4	0.02	99.98

Our results: there is no isospin vector states

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J^P	I	Mass	Width	E_B	$\mathcal{P}_{BB}$	$\mathcal{P}_{B^*B^*}$	Γ_{BB}	$\Gamma_{B^*B^*}$
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		10640.7	2.8	8.7	76%	24%	2.8	0
	1	10545.9	0	13.1	93%	7%	0	0
		10672.6	72.0	−23.2	39%	61%	30.7	41.3
2^+	1	10642.3	0	7.1	-	100%	-	0

The S-wave BB states can not be $J^P(I) = 0^+(0)$

Fermion sign problem in DMC

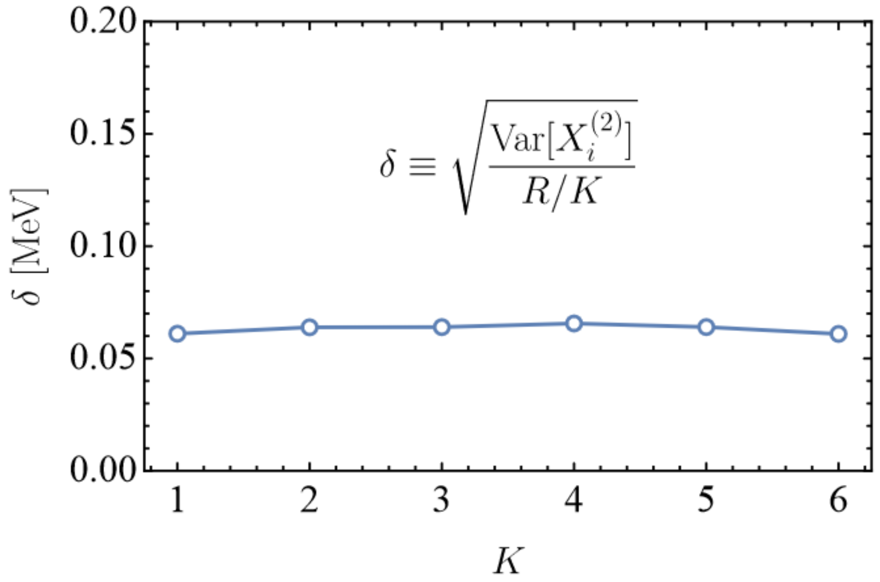
- The density of the walker is always positive
- In principle, the mathematical ground state of few-body system has no node
- Considering the identical particles, the Antisymmetrization of the fermions introduce the nodes
- For boson systems, the excited state wave functions have nodes
- How to get $\psi(R)$ with negative part?
- Naive strategy:
  - ▶ $\psi(R) = \psi^+(R) - \psi^-(R)$, $\psi^+(R) > 0$ and $\psi^-(R) > 0$
  - ▶ Give each walker a label: (+) or (-) to sample $\psi^+(R)$ and $\psi^-(R)$ respectively
  - ▶ $\psi^+(R)$ and $\psi^-(R)$ will approach to the same mathematical ground state $\psi_0(r)$
  - ▶ Significant cancellation!!! Large noise!!! Fermin sign problem
- Fix node: kill the walker across the nodal surface

Statistical uncertainties of DMC

- Jackknife resampling method

$$\sigma[\bar{X}] = \sqrt{\frac{1}{R(R-1)}\sum_i^R (X_i^{(1)} - \bar{X})^2}$$
$$= \sqrt{\frac{R-1}{R}\sum_i^R (\bar{X}_{(i)\text{jack}} - \bar{X}_{\text{jack}})^2}\,.$$

- Statistical uncertainties: less than 1 MeV

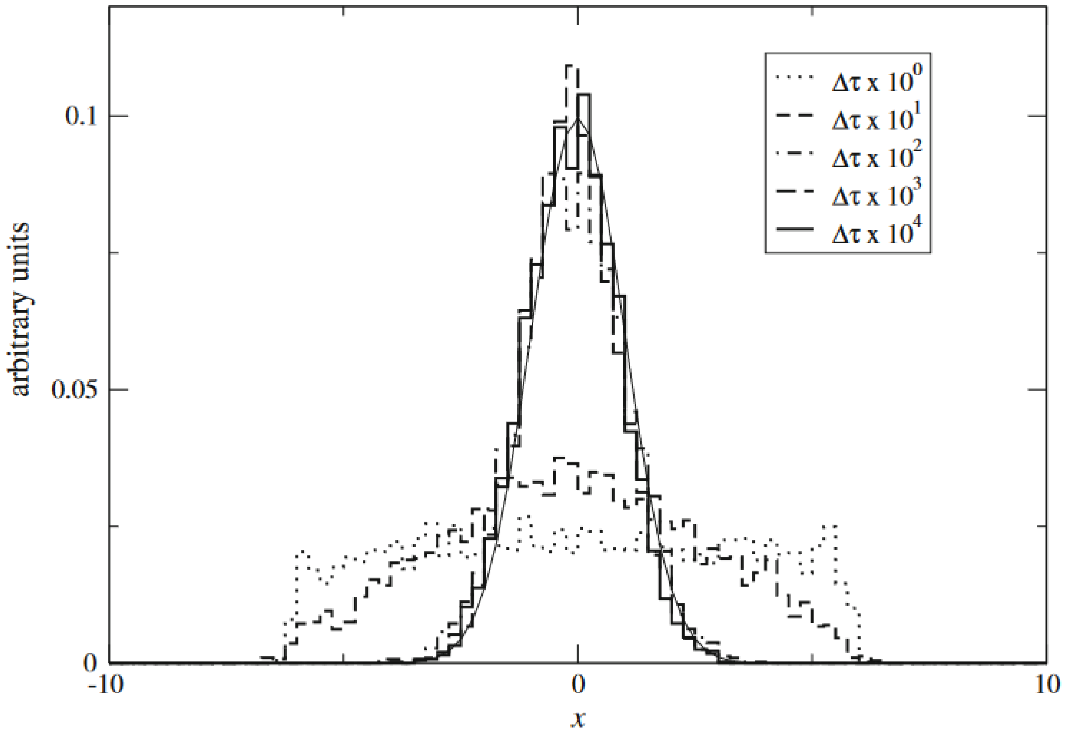


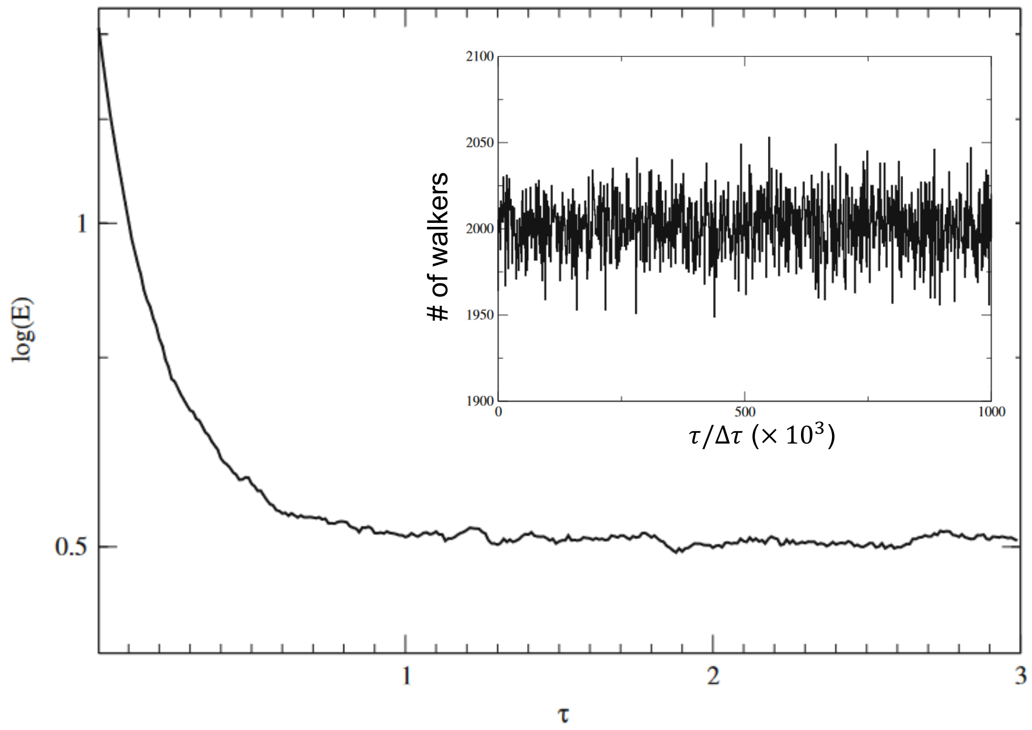
A simple example of Naïve DMC

- One-dimensional HO: $H = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^2$, $E_0 = 0.5$
- Initial wave functions: a constant
- Clear exponential decay of the energy towards the exact E_0
- The # of walkers depart from the central value by more than 3%

$$n_r = \text{Floor}\left[e^{-\left(\frac{V(R)+V(R')}{2} - E_R\right)\Delta\tau} + u\right]$$

Walkers where the potential changes drastically, lead to large fluctuations of the population





Importance sampling

- In the practical simulation, the ψ_T is unknown beforehand
- Two-body Jastow correlation factor

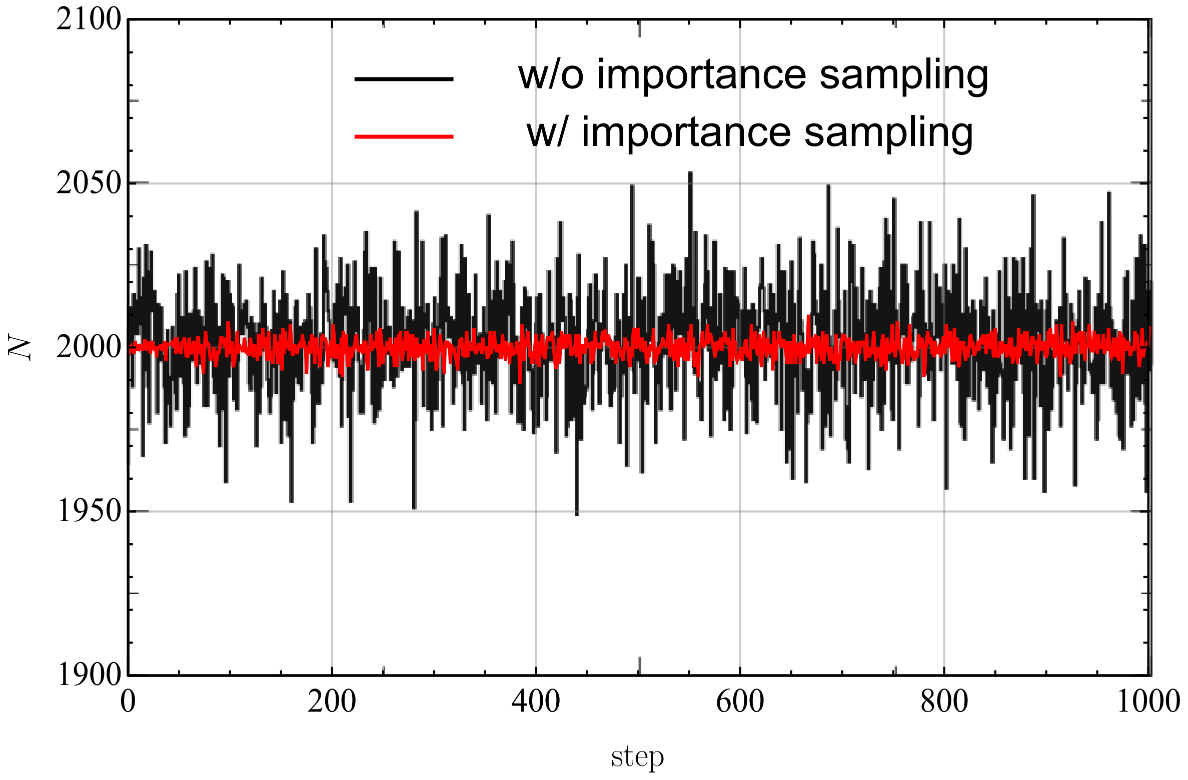
$$\psi_T(\boldsymbol{R}) = \prod_{i<j} \exp\left(\frac{a_{ij} r_{ij}}{1+\beta_{ij} r_{ij}}\right)$$

- In our calculation

$$\psi_T(\boldsymbol{R}) = \prod_{i<j} e^{-a_{ij} r_{ij}}$$

  - ► a_{ij} are adjustable constants to minimize the fluctuation

- With importance sampling, the fluctuation is reduced



Coupled-channel in DMC

- Coupled channels

$$\Psi(\boldsymbol{R},t) = \sum_\alpha \Psi_\alpha(\boldsymbol{R},t)\chi_\alpha\,,$$

$$-\frac{\partial\Psi_{\alpha'}}{\partial t} = \sum_\alpha \hat{H}_{\alpha'\alpha}\Psi_\alpha - E_R\Psi_{\alpha'}\,.$$

- Sampling $\mathcal{F}$

$$f_\alpha(\boldsymbol{R},t) \equiv \psi_T(\boldsymbol{R})\Psi_\alpha(\boldsymbol{R},t)\,,$$

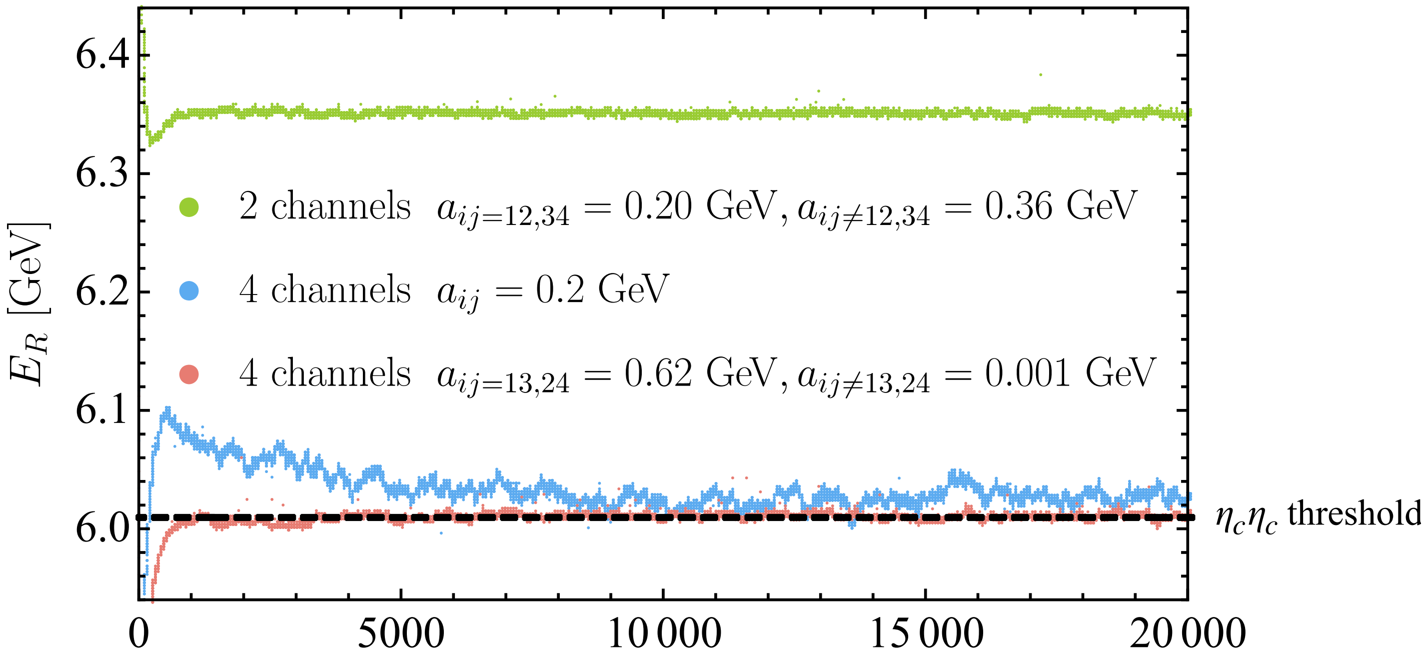
$$\mathcal{F}(\boldsymbol{R},t) \equiv \sum_\alpha f_\alpha(\boldsymbol{R},t)\,.$$

- Assuming $\mathcal{F}$ is positive such that can be sampled by distribution of walkers

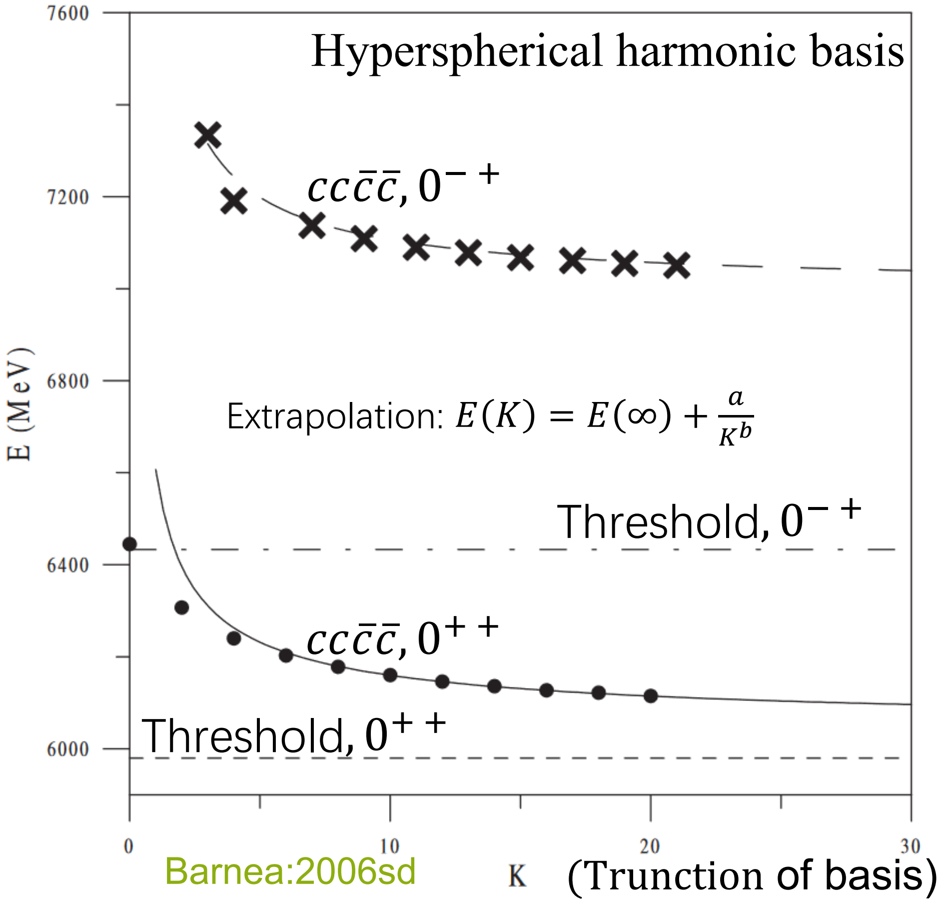
Gordillo:2020sgc

DMC in quark models

- Get a di-meson type ground state without presumed such kind of correlation
  - ▶ The importance wave function with even two-body correlations give the di-meson thresh.
    $a_{12} = a_{13} = a_{23} = a_{14} = a_{23} = a_{34}$
- In variational method, it is hard to get the di-meson without the di-meson type in basis
- No need presumed clustering behaviors!!!



$$\psi_T(\boldsymbol{R}) = \prod_{i<j} e^{-a_{ij}r_{ij}} .$$



Barnea:2006sd