



东南大学 | 25.06.2024

Extracting hadron potentials from HAL QCD wave functions

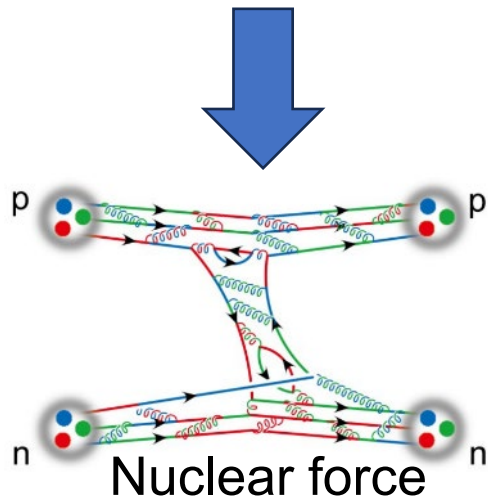
Lu Meng (孟璐) | RUHR-UNIVERSITÄT BOCHUM

Based on papers in preparation
Together with Evgeny Epelbaum (RUB)

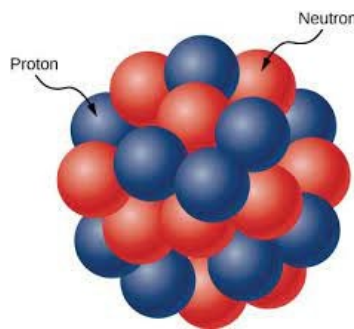
From QCD to nuclei and neutron stars

$$\text{QCD: } \mathcal{L}_{QCD} = \sum_f \bar{q}_f (i\not{D} - \mathcal{M}_{q_f}) - \frac{1}{4} G_{\mu\nu}^a G^{\mu\nu,a}$$

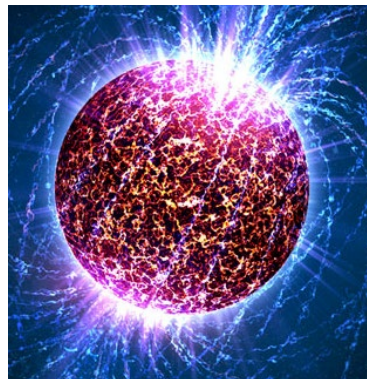
Phenomenological
model



Chiral effective
field theory



nucleus

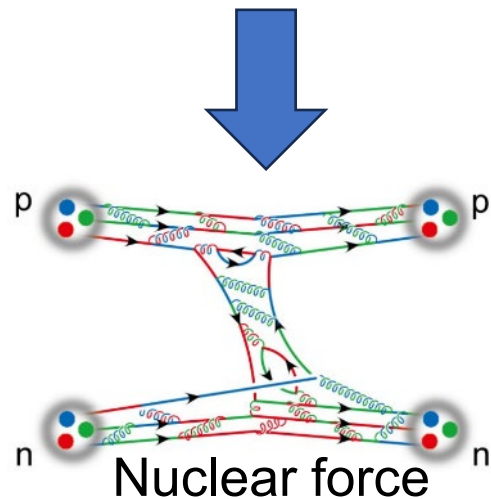


neutron star

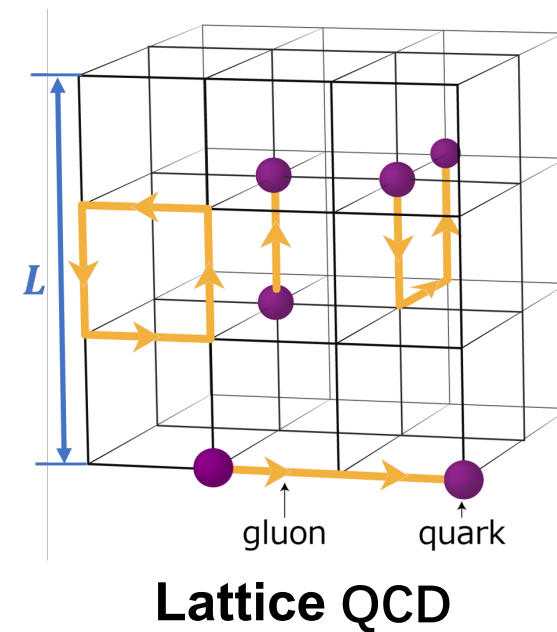
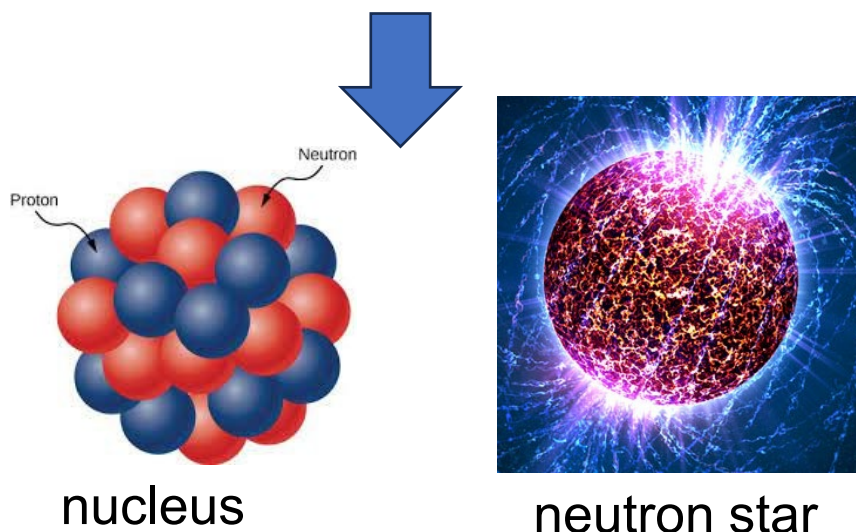
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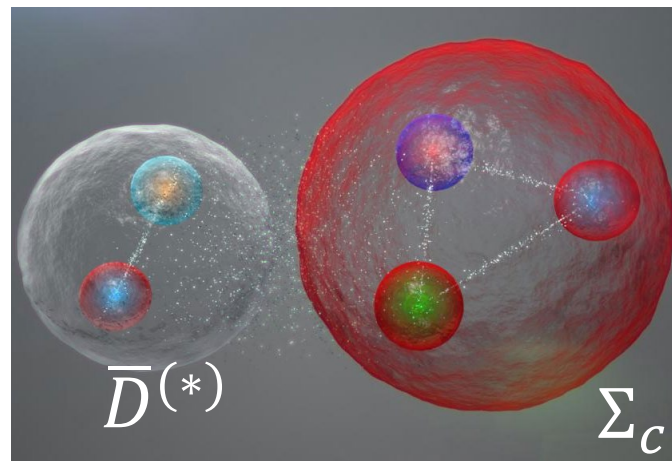


Chiral effective field theory



Hadronic molecule

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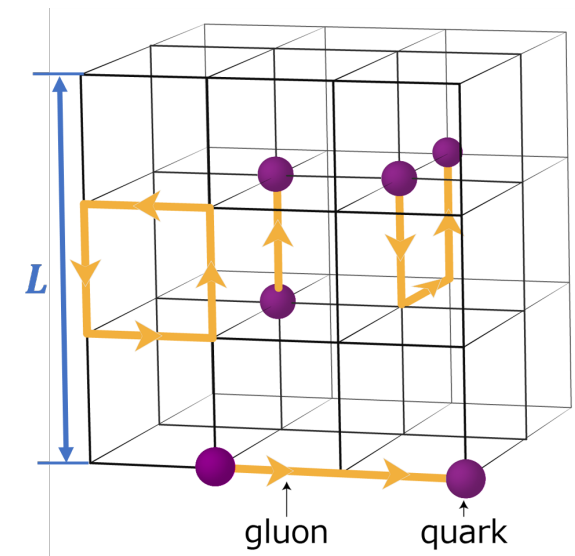
hadronic molecule

LHCb:2015yax

Phenomenological
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Chiral effective
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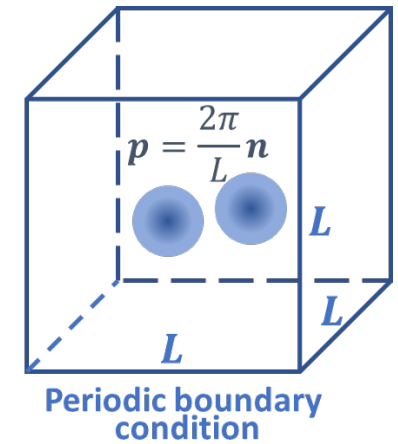
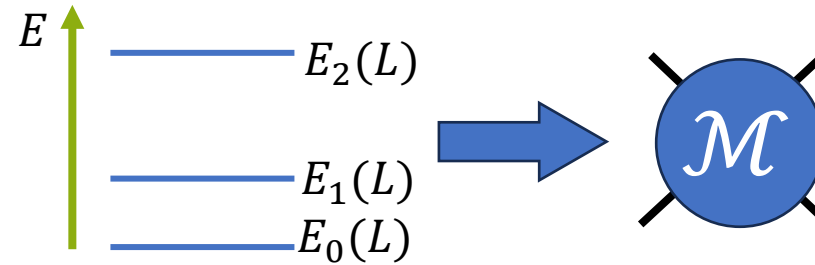


Lattice QCD

Energy level method

- Raw data: finite volume energy levels E^{FV}
- Get observables: Lüscher's formula
 - ▶ $E^{FV} \sim \delta(E^{FV})$

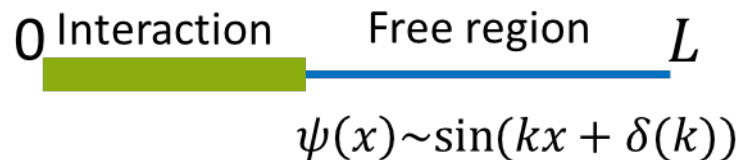
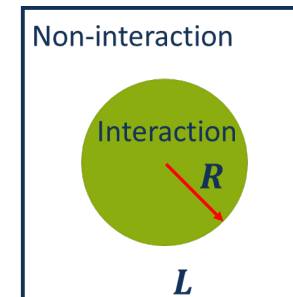
Lüscher:1990ux
AKA: Lüscher method



- Requirement: larger box and short interaction $\frac{L}{2} \gg R$

- Derivation example of Lüscher's formula

- ▶ The $(\nabla^2 + k^2)\psi(r) = 0$ for $r > R$
- ▶ Asymptotic behavior: $\psi_k(r) \sim \frac{e^{i\delta(k)} \sin[kr + \delta(k)]}{kr}$
- ▶ 1-D example

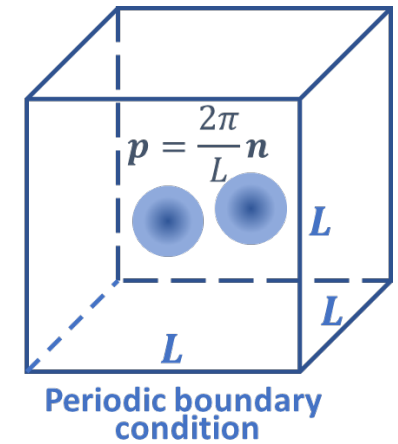
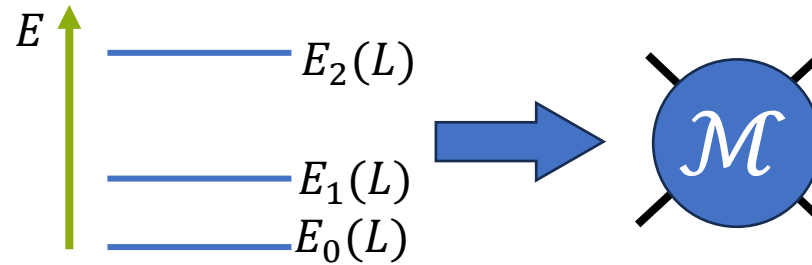


- ▶ Quantization condition: $\sin(kL + \delta(k)) = 0 \Rightarrow kL + \delta(k) = n\pi$

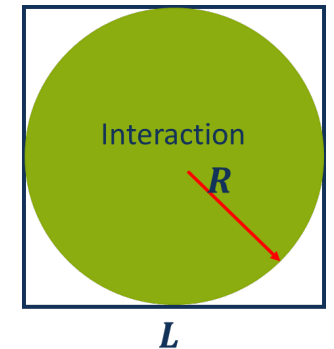
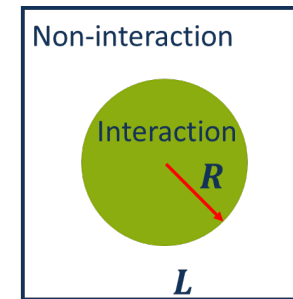
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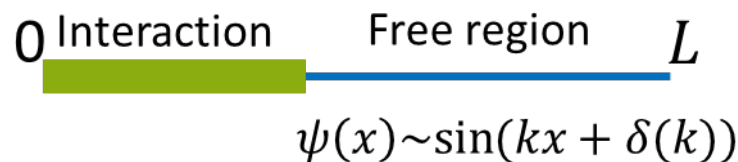
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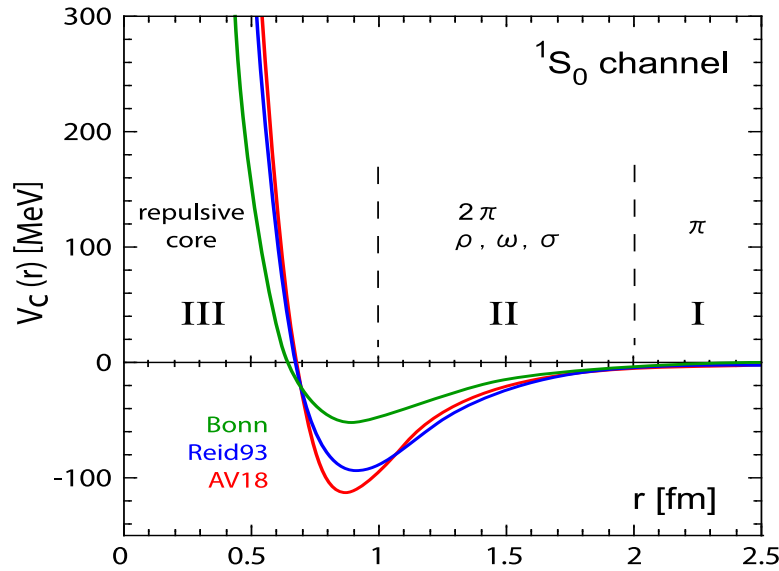
Long-range interaction and small box
My talk in 强子在线论坛



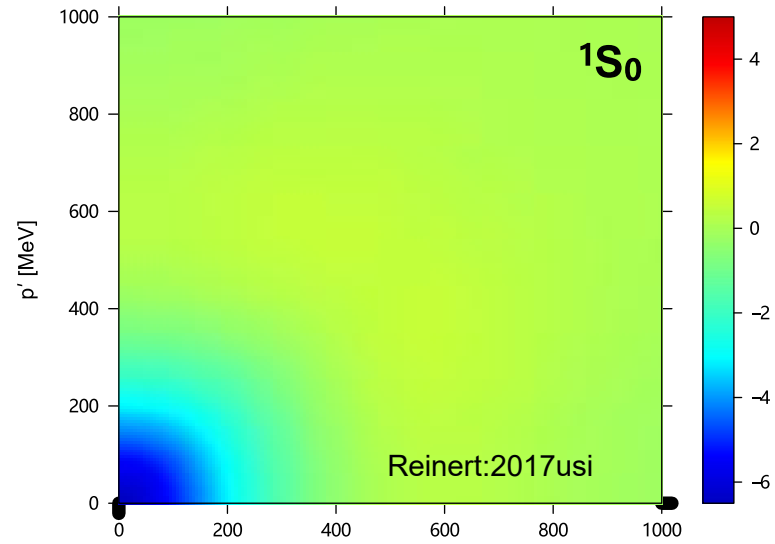
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[JHEP10\(2021\)051](#), and [arXiv:2312.01930](#)

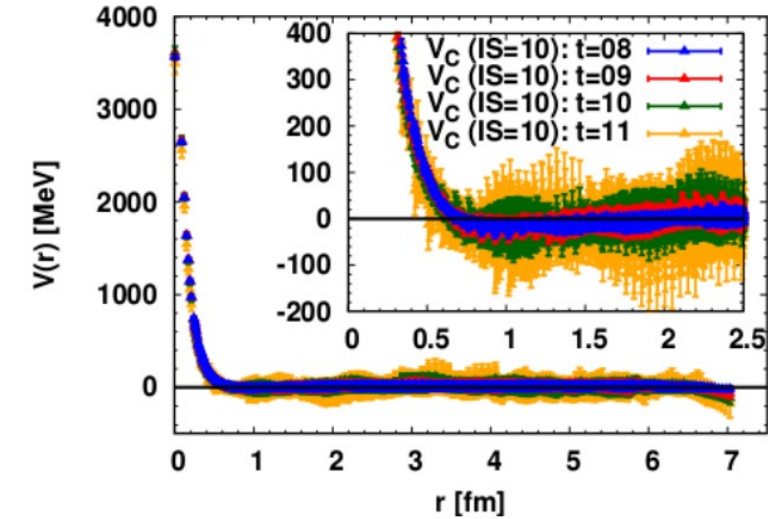
Potential methods



Phenomenological model



Chiral EFT^p [MeV]



HAL QCD

- Raw data: Nambu-Bethe-Salpeter wave function (NBS WFs) [Ishii:2006ec,Aoki:2009ji,Aoki:2012tk](#)
- Get observables AKA: HAL QCD (Hadrons to atomic nuclei from LQCD) method



- ▶ Often criticized for **uncontrolled systematics**
- ▶ Potentials cannot be determined by the observables (phase shifts etc)

To bind or not to bind

bind in both 3S_1 and 1S_0 channels

● With (deeply) bound NN

● Without bound NN (or inconclusive)

1 2006 NPLQCD First dynamical calculations

2011 NPLQCD $M_\pi \approx 390$ MeV

2012 Yamazaki et al. $M_\pi \approx 510$ MeV

2015 NPLQCD $M_\pi \approx 800$ MeV

2015 Yamazaki et al. $M_\pi \approx 310$ MeV

2 2015 CalLat $M_\pi \approx 800$ MeV+P,D,F waves

2015 NPLQCD $M_\pi \approx 450$ MeV

2020 NPLQCD $M_\pi \approx 450$ MeV

2012 HAL QCD $M_\pi \approx 710$ MeV

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3 2019 “Mainz” $M_\pi \approx 960$ MeV

2020 CoSMoN $M_\pi \approx 714$ MeV

2021 NPLQCD $M_\pi \approx 800$ MeV

□ However, we are observing a **preponderance of evidence** that the older methods with present statistics, are yielding qualitatively incorrect spectrum —

I believe the old results are wrong (including those I was involved with)

I believe the di-nucleon system unbinds at pion masses heavier than physical

Talk of A.Walker-Loud in lattice2023:<https://indico.fnal.gov/event/57249/contributions/271301/>

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Uncontrolled systematics

Significance of the HAL QCD method
To improve the understanding of the systematics of HAL QCD

Talk of A. Walker-Loud in lattice2023: <https://indico.fnal.gov/event/57249/contributions/271301/>

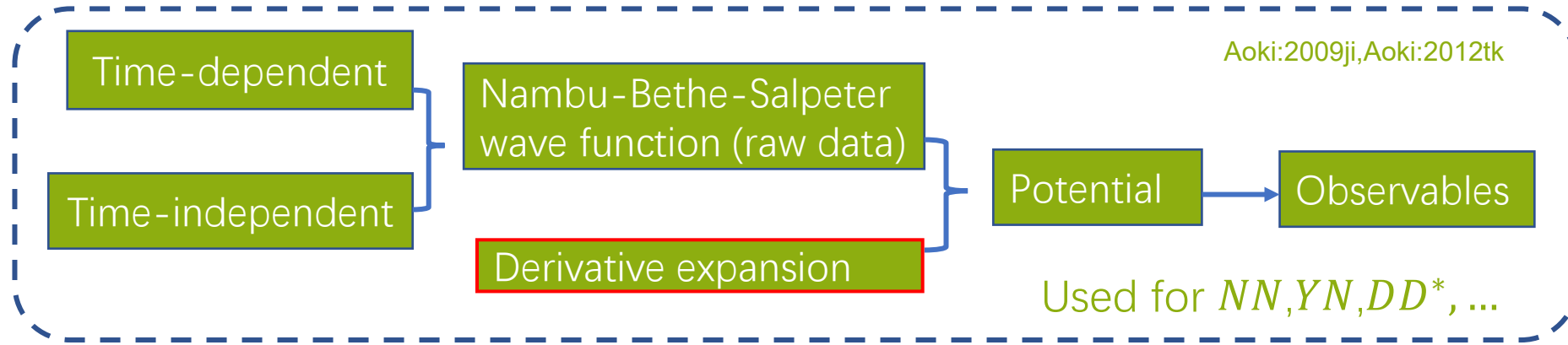
HAL QCD method

Disclaimers:

- I am not the member of the HAL QCD group
- I will try my best to be fair

Aoki:2009ji,Aoki:2012tk

HAL QCD method



- The raw data of HAL QCD simulations are Nambu-Bethe-Salpeter (NBS) wave functions
- The derivative expansion (DE) method is often questioned by some people
 - ▶ Converge fast for NN
 - ▶ Converge slowly for $\pi\pi$
- In this talk, I will first illustrate some concepts and then provide an alternative way of DE method

- The equal-time BS amplitude (BS wave function, BSWF)

CP-PACS:2005gzm

$$\psi(\vec{x}; \vec{k}) = \langle 0 | \pi_1(\vec{x}/2) \pi_2(-\vec{x}/2) | \pi_1(\vec{k}), \pi_2(-\vec{k}); in \rangle$$

- Asymptotic behavior of BS wave function

$$\psi(\vec{x}; \vec{k}) = e^{i\vec{k} \cdot \vec{x}} + \int \frac{d^3 p}{(2\pi)^3} \frac{T(p; k)}{p^2 - k^2 - i\epsilon} e^{i\vec{p} \cdot \vec{x}}$$

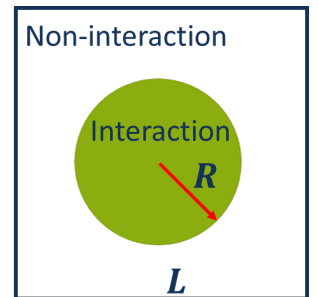
- ▶ $T(p; k)$ is the half-on-shell T-matrix
 - ▶ $\psi(\vec{x}; \vec{k})$ satisfy the Lippmann-Schwinger eq. as the non-relativistic scattering wave function
 - ▶ Using Nishijima, Zimmermann and Haag (NZH) reduction formula for composite local operators
 - ▶ No non-relativistic approximation, just a formal resemblance
- The BSWF at different energies $\{k_i\}$ in the lattice are the raw data of t-independent HAL QCD
 - The general problem: $\psi_{k_i}(\vec{x}) \Rightarrow V$ from Schrodinger-like equations

Time-independent HAL QCD

- The general problem (set $m = 1$, 1D case as an example)

$$\int dr' V(r, r') \psi_{k_i}(r') = \left(\frac{d^2}{dr^2} + k_i^2 \right) \psi_{k_i}(r) \Rightarrow \int dr' V(r, r') R^{(i)}(r) = K^{(i)}(r)$$

- ▶ Determined the potential $V(r, r')$ once $\{\Psi_{k_i}(r)\}$ are given
 - ▶ $R^{(i)}(r)$ and $K^{(i)}(r)$ are known
 - ▶ **Note: the # of wave functions is small, 2 or 3**
- In general, the potential is **nonlocal**. No reason to reject the nonlocal potentials
 - ▶ Definition of Local potential: $V(r, r') = V(r)\delta(r - r')$
- Regions of potential
 - ▶ Inner region (interacting region): $V(r, r') \neq 0$ ($r, r' < R$)
 - ▶ outer region (asymptotic region): $V(r, r') = 0$ ($r, r' > R$)
 - ▶ The raw data is $\psi_{k_i}(\vec{x})$ in the **interacting region +** outer region
 - ▶ In principle, one can get the $\delta(k_i)$ from $\psi_{k_i}(\vec{x})$
Asymptotic properties
 - ▶ The Lüscher's method only concerns on the asymptotic region
 - ▶ Could we get **more information than $\delta(k_i)$** from the $\psi_{k_i}(\vec{x})$?



Time-dependent HAL QCD

- ψ_{k_i} with fixed energies are projected from the correlation function after ground state saturation

$$R(r, t) = \sum_n a_n \psi_{k_n}(r) e^{-(2\sqrt{m_N^2 + k_n^2} - 2m_N)t}$$

Ishii:2012ssm

- For large box, it is expensive to get the ground state saturation
- Time-dependent Schrödinger-type equation

$$\left(-\frac{\partial}{\partial t} + \frac{1}{4m_N} \frac{\partial^2}{\partial t^2} \right) R(r, t) = \left(\hat{H}_0 + \hat{V} \right) R(r, t)$$

- Time-dependent strategy without ground state saturation makes simulations with large box and small pion mass available

▶ $m_\pi = 146$ MeV, $a \simeq 0.0846$ fm, $L^4 = 96^4$, $L = 8.1$ fm

Doi:2017zov, Lyu:2022imf, Lyu:2023xro...

- The general problem

$$\int dr' V(r, r') R(r', t) = K(r, t) \quad K(r, t) = \left(-\frac{\partial}{\partial t} + \frac{1}{4m_N} \frac{\partial^2}{\partial t^2} - \frac{1}{m_N} \frac{d^2}{dr^2} \right) R(r', t)$$

- **Note: the # of wave functions is small, 2 or 3**

Modern views of potential

Local potential VS nonlocal potential

- There is no reason to rule out the nonlocal potential either in principle or phenomenologically Bogner:2009bt
- Potential is not observable
 - ▶ Cannot be determined uniquely by scattering experiments
 - ▶ Observable-equivalent potentials are related by unitary trans. (UT) or field redefinition
$$H|\psi\rangle = E|\psi\rangle \Rightarrow UHU^\dagger U|\psi\rangle = EU|\psi\rangle \Rightarrow \tilde{H}|\tilde{\psi}\rangle = EU|\tilde{\psi}\rangle$$
 - ▶ UT can relate local potentials to nonlocal potentials Ekstein:1960xkd
- Inverse scattering problem: given phase shifts, find the potential
 - ▶ To make the potential unique: local potential is assumed (no physical reason)
- Local potentials are an advantage or even a necessity for some methods of solving the nuclear many-body problem.
- There are still methods for nonlocal potentials: no-core shell model
- Sometimes, the nonlocal potential are advantage for
 - ▶ Separable interaction: $V = v_{ij}|i\rangle\langle j|$, or $V(r, r') = v_{ij}\varphi_i(r)\varphi_j(r')$ for the three body problem (Fadeev function)

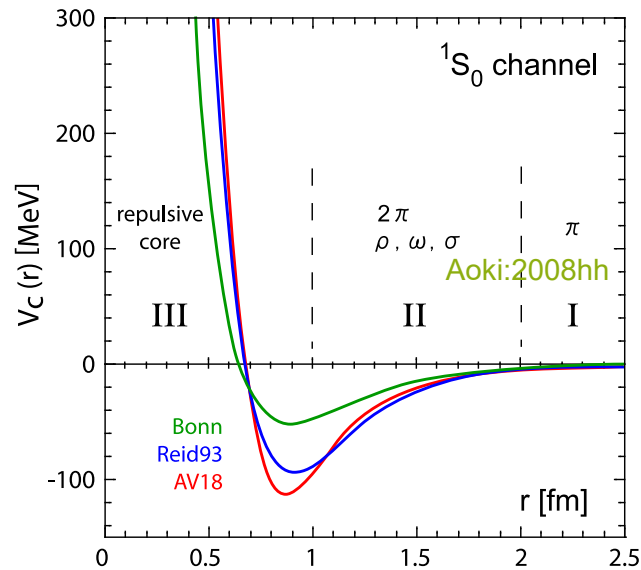
Phase-equivalent potentials

- Different nuclear forces are phase-equivalent potentials
- Similarity renormalization group (SRG) are used to soften the short-range interaction and increase the convergence of the many-body problems

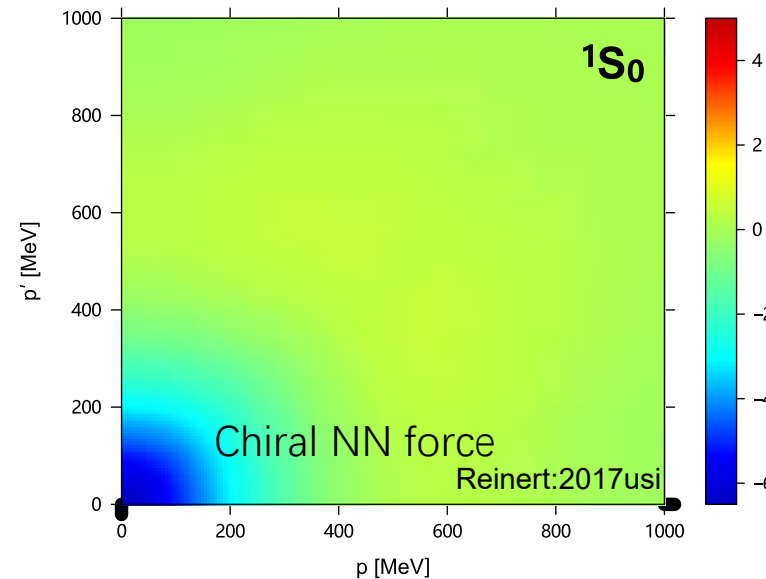
► $H_s = U(s)HU^\dagger(s)$, choose a specific $U(s)$ gives the flow equation

$$\frac{dH_s}{ds} = [[T_{rel}, H_s], H_s]$$

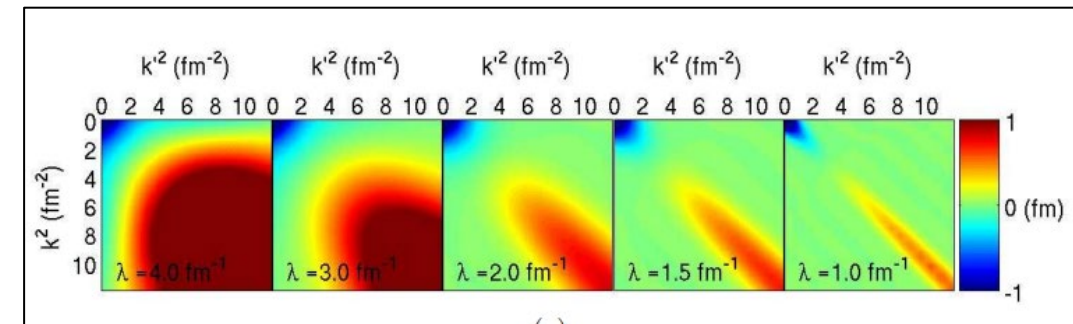
Bogner:2009bt



Local NN force



Nonlocal NN force



SRG evolution

- Non-observables

- ▶ Non-asymptotic behavior of ψ , e.g. the deuteron D-state probability

Amghar:1995av

- ▶ Off-shell T-matrix

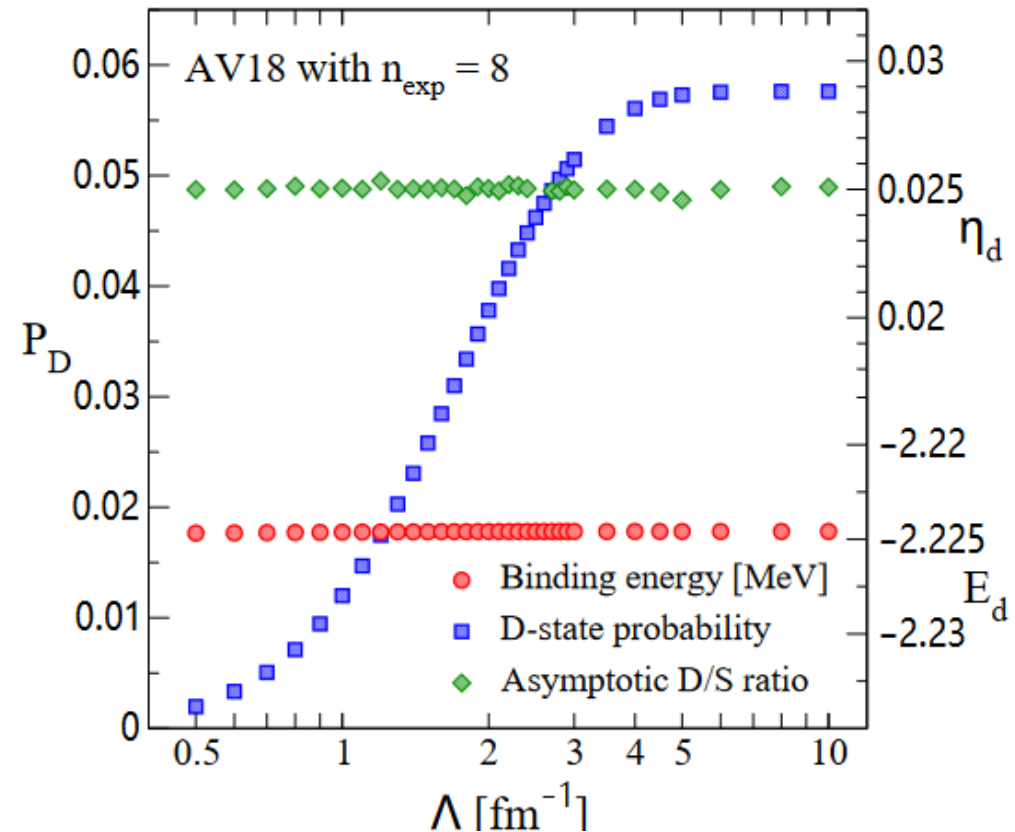
- ▶ Potential

- Observables

- ▶ Asymptotic behavior of ψ

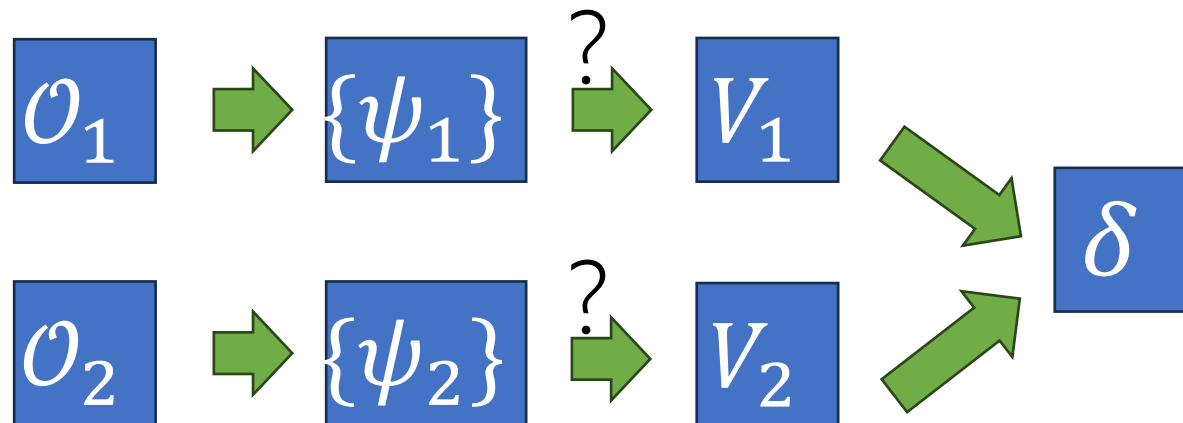
- ▶ Phase shift

- ▶ On-shell T-matrix



Interpolating operator VS potential

- In principle one may choose any composite operators with the same quantum numbers as the hadron to define the BS wave function
- Different operators give different BS wave functions and different hadron potentials
 - ▶ They are related by UT
 - ▶ We anticipate they lead to the same observables such as the δ and E_b
- **In the HAL QCD simulations: once the setting of interpolating operators (its smearing) are fixed, the “underlying” potential is fixed in principle**
- The “underlying” potential cannot be extracted from only a small number of the wave functions

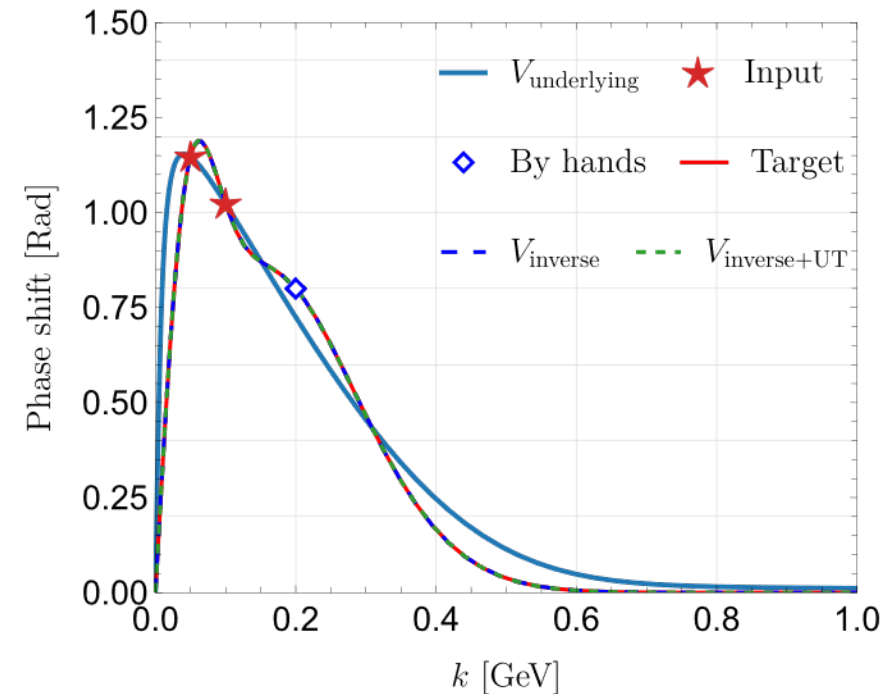
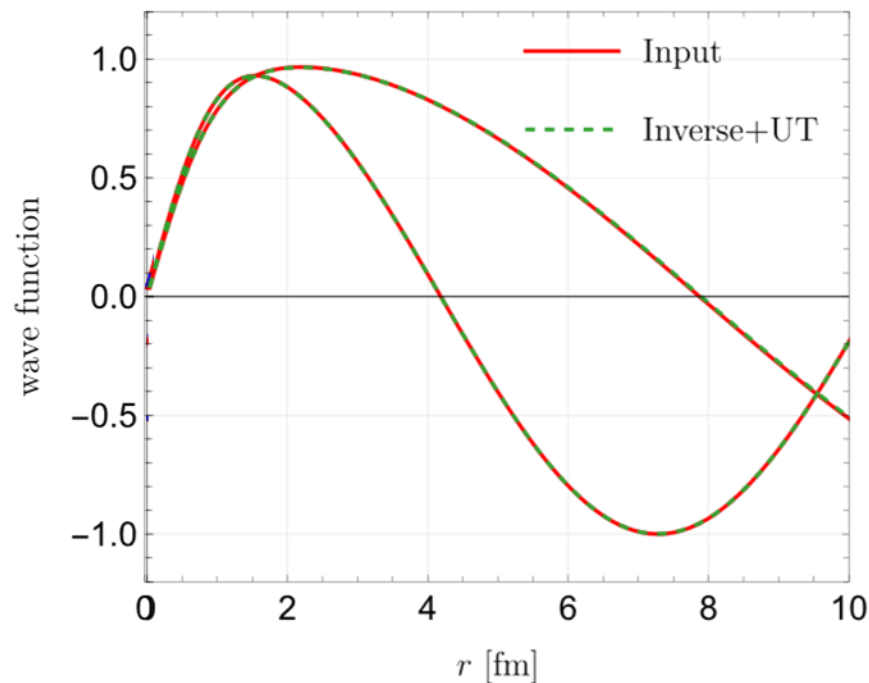


Interpolating operator VS potential

- From a small number of the wave functions, the potential can not be determined uniquely
 - ▶ Think it in a discrete way

$$\int dr' V(r, r') R^{(i)}(r) = K^{(i)}(r) \Rightarrow \mathbb{V}_{N \times N} R_{N \times 1}^{(i)} = K_{N \times 1}^{(i)}$$

- ▶ One need N wave functions to fix potential matrix $\mathbb{V}_{N \times N}$
- ▶ N: several tens, typical order of # quadrature points
- ▶ In practices, only 2 or 3 wave functions are accessible

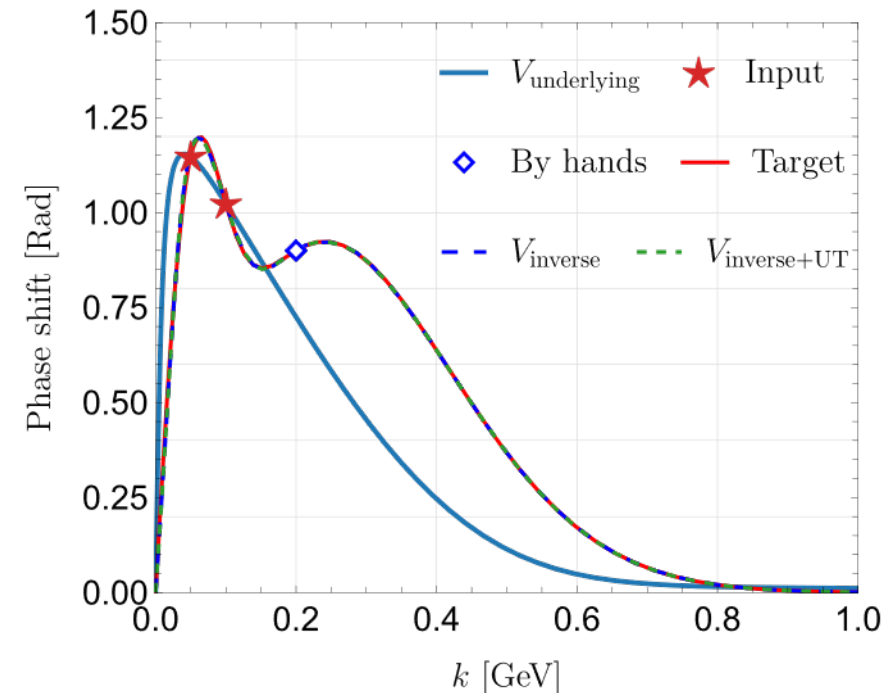
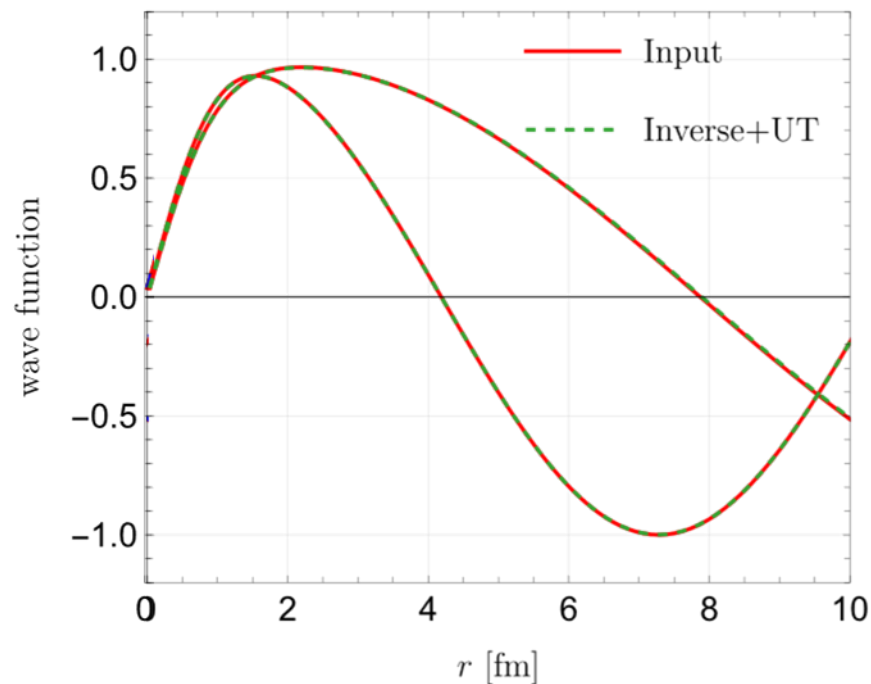


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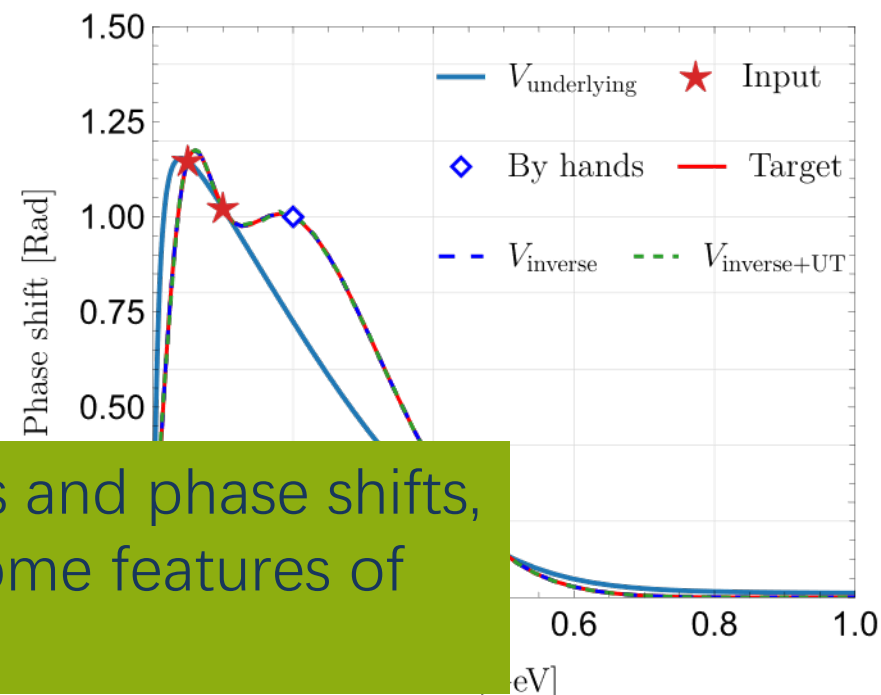
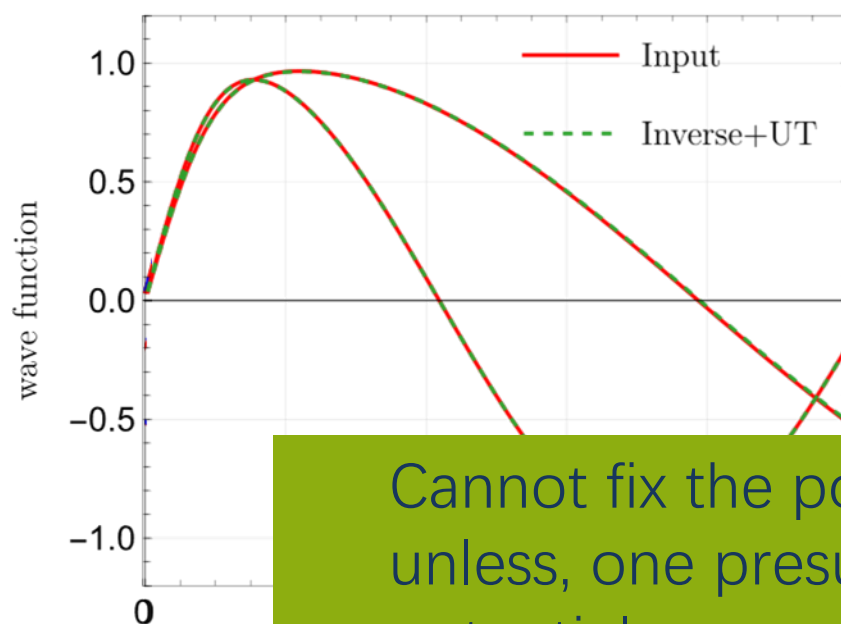


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Cannot fix the potentials and phase shifts, unless, one presumes some features of potentials

Derivative expansion VS EST expansion

Derivative expansion

- Derivative expansion

Aoki:2021ahj

$$V(r, r') = V_0(r)\delta(r - r') + V_1(r)\delta(r - r')\frac{d^2}{dr'^2} + V_2(r)\delta(r - r')\frac{d^4}{dr'^4} + \dots$$

- LO

$$V_0(r)R^{(1)}(\vec{r}) = K^{(1)}(\vec{r}) \Rightarrow V_0(r) = \frac{K^{(1)}(\vec{r})}{R^{(1)}(\vec{r})}$$

- NLO

$$\begin{pmatrix} R^{(1)}(r) & \frac{d^2}{dr^2}R^{(1)}(r) \\ R^{(2)}(r) & \frac{d^2}{dr^2}R^{(2)}(r) \end{pmatrix} \begin{pmatrix} V_0(r) \\ V_1(r) \end{pmatrix} = \begin{pmatrix} K^{(1)}(r) \\ K^{(1)}(r) \end{pmatrix}$$

- It is not expansion about some definite small quantities

- Its convergence is tested self-consistently

- Think it in a discrete way,

$$\frac{d^2}{dr^2}\psi(x_n) \approx \frac{\psi(x_{n-1}) + \psi(x_{n+1}) - 2\psi(x_n)}{h^2}$$

$$V_0 = \left[\begin{array}{c} \diagdown \\ \diagdown \\ \diagdown \end{array} \right], V_1 = \left[\begin{array}{c} \diagdown \diagdown \\ \diagdown \diagdown \\ \diagdown \diagdown \end{array} \right], V_2 = \left[\begin{array}{c} \diagdown \diagdown \diagdown \\ \diagdown \diagdown \diagdown \\ \diagdown \diagdown \diagdown \end{array} \right], \dots$$

► The band width become wider

Singular potential

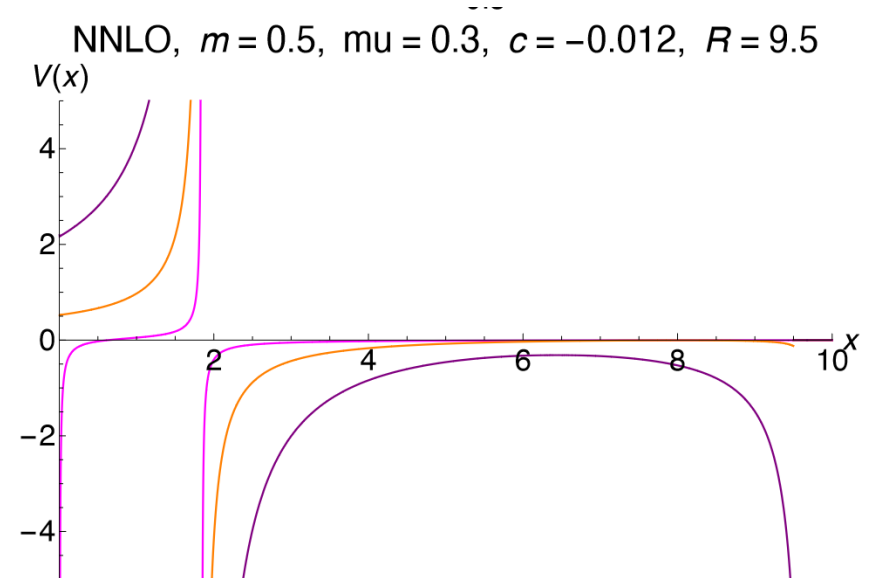
- NLO derivative expansion

$$\begin{pmatrix} R^{(1)}(r) & \frac{d^2}{dr^2} R^{(1)}(r) \\ R^{(2)}(r) & \frac{d^2}{dr^2} R^{(2)}(r) \end{pmatrix} \begin{pmatrix} V_0(r) \\ V_1(r) \end{pmatrix} = \begin{pmatrix} K^{(1)}(r) \\ K^{(1)}(r) \end{pmatrix}$$

- The potential become singular at the zero of det of the coefficients matrix

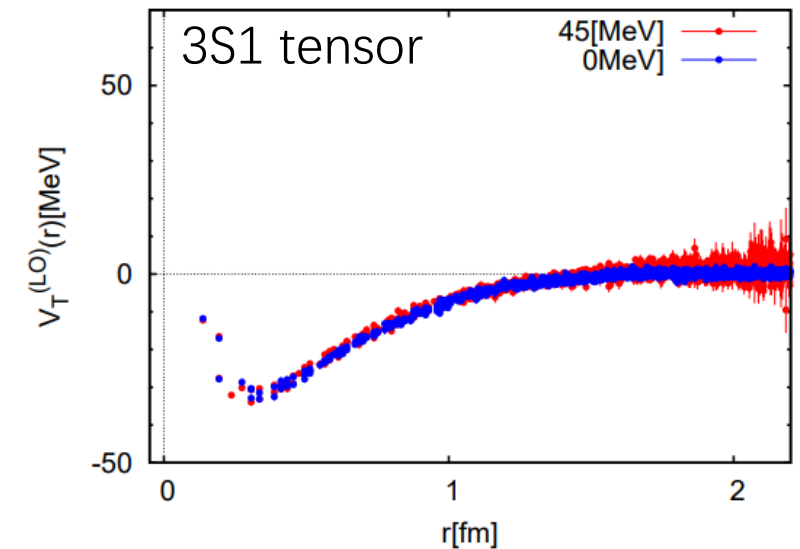
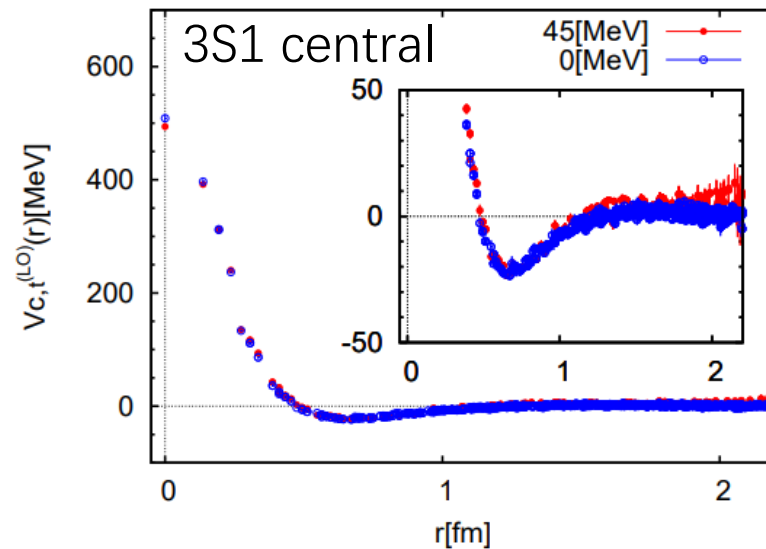
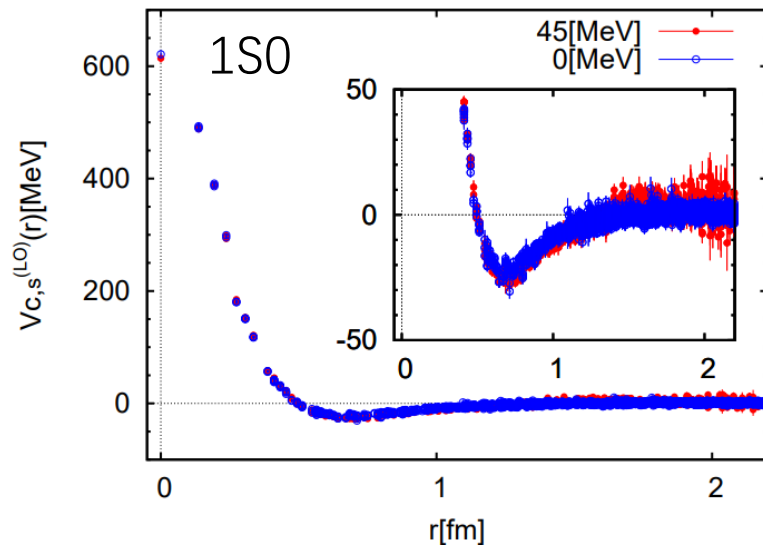
- A example from toy model [Aoki:2021ahj](#)

- ▶ In simulation, it is challenging to handle the singularity
- ▶ Wave functions are obtained at discrete point.



Locality of the potential

- Self-consistence test: LO NN potentials obtained at different energies ($E \simeq 0$ MeV and 45 MeV)
 - ▶ LO approximation of DE validates to $E = 45$ MeV. [Murano:2011nz](#)
 - ▶ Other test: optimized operators method [Lyu:2022tsd](#)



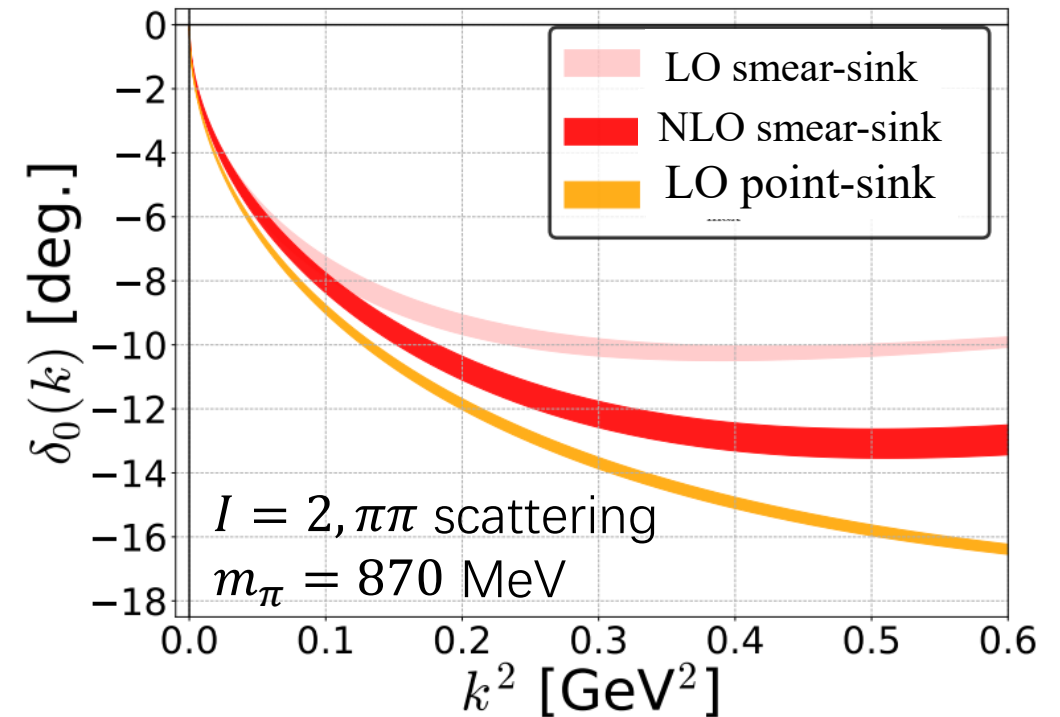
- Point-sink scheme used by HAL QCD group
 - ▶ Very local “underlying” potential, fast convergence of the DE

By experiences

[Murano:2011nz](#), [HALQCD:2017xsa](#), [Kurth:2013tua](#)

- In the case of two-particle scattering processes involving quark annihilation diagrams
 - ▶ smear-sink scheme
 - ▶ DE method does not converges as fast as point-sink scheme
 - ▶ The “underlying” potentials of the are more non-local than those of point-sink scheme

HALQCD:2017xsa



- To solve this problem, the HAL QCD group has made extensive efforts to improve numerical computation methods while retaining the DE method. Akahoshi:2019klc, Akahoshi:2021sxc
- Local potentials do not possess any essential superiority over a non-local potentials.
- Perhaps, turning to another parameterization of the potentials will take less pains.
 - ▶ Separable parameterization

Separable representation

- The problem: $V|R^{(i)}\rangle = |K^{(i)}\rangle$
- Ernst-Shakin-Thaler (EST) method:

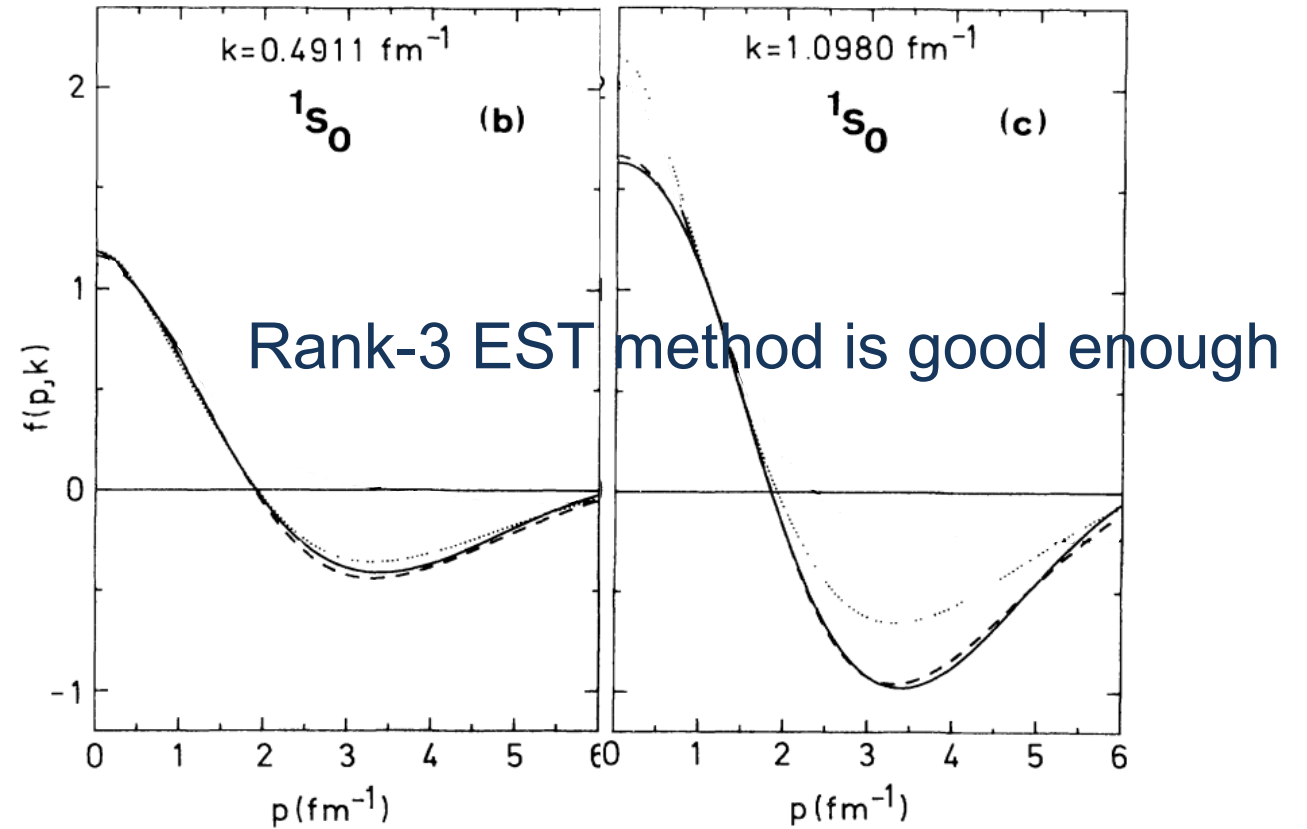
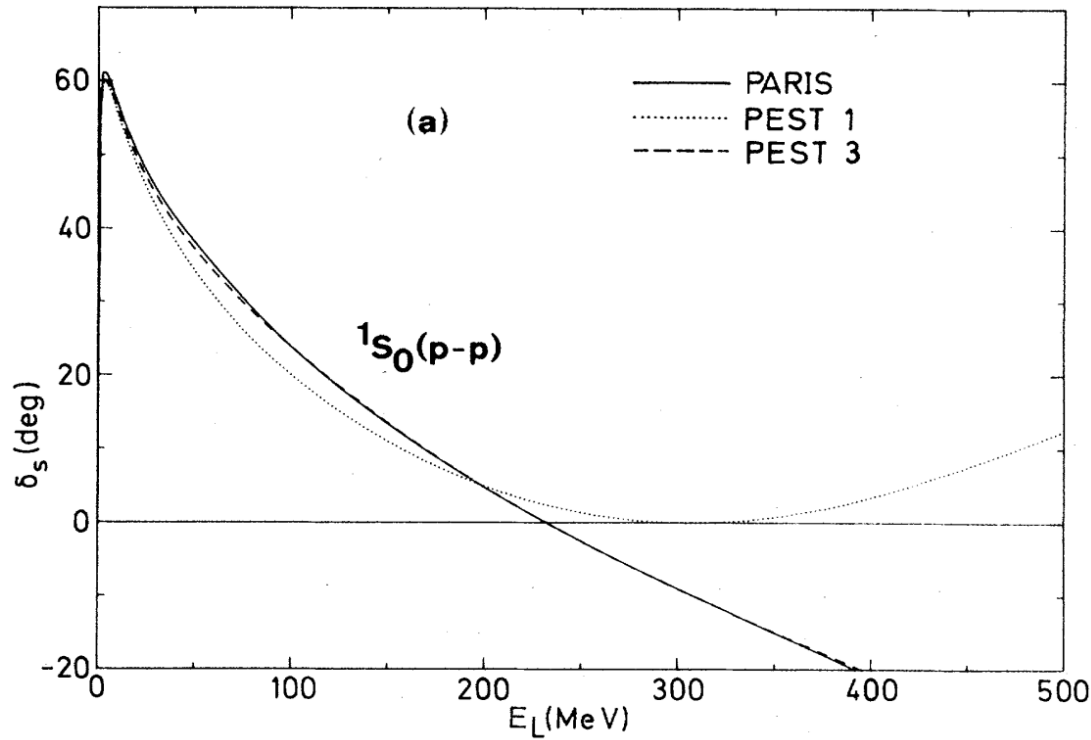
$$V = \sum_{mn} |K^{(m)}\rangle \Lambda_{mn} \langle K^{(n)}|, \quad \Lambda_{mn} \langle K^{(n)}|R^{(i)}\rangle = \delta_{mi}$$

- ▶ To solve the Faddeev function, find the equivalent separable potential of the high-precision nuclear force
- ▶ on-shell and off-shell equivalent

Ernst:1973zzb,Haidenbauer:1984dz

Separable representation

► Application: on-shell and off-shell equivalent separable potentials of Paris potentials



Ernst:1973zzb,Haidenbauer:1984dz

Numerical comparisons

Two underlying potentials

- Separable potential Aoki:2021ahj

$$V(\mathbf{r}, \mathbf{r}') = \omega \frac{e^{-\mu r}}{r} \frac{e^{-\mu r'}}{r'}$$

- LO chiral nuclear force Reinert:2017usi

$$V_{ctc}(\mathbf{p}, \mathbf{p}') = C e^{-\frac{p^2 + p'^2}{\Lambda^2}}, \quad V_{ope}(\mathbf{q}) = -\frac{g_A}{4F_\pi^2} \left(\frac{\boldsymbol{\sigma}_1 \cdot \mathbf{q} \boldsymbol{\sigma}_2 \cdot \mathbf{q}}{q^2 + m_\pi^2} + C_{sub} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \right) e^{-\frac{q^2 + m_\pi^2}{\Lambda^2}}$$

- ▶ Separable contact interaction + local one-pion exchange interaction
- For simplicity: S-wave and 1S_0 NN interaction
- Solve the Time-(in)dependent Schrodinger equation to get wave functions
- Time-independent method
 - ▶ Choose $\{\psi_{k_i}\}$ as inputs
- Time-dependent method
 - ▶ Initial wave functions

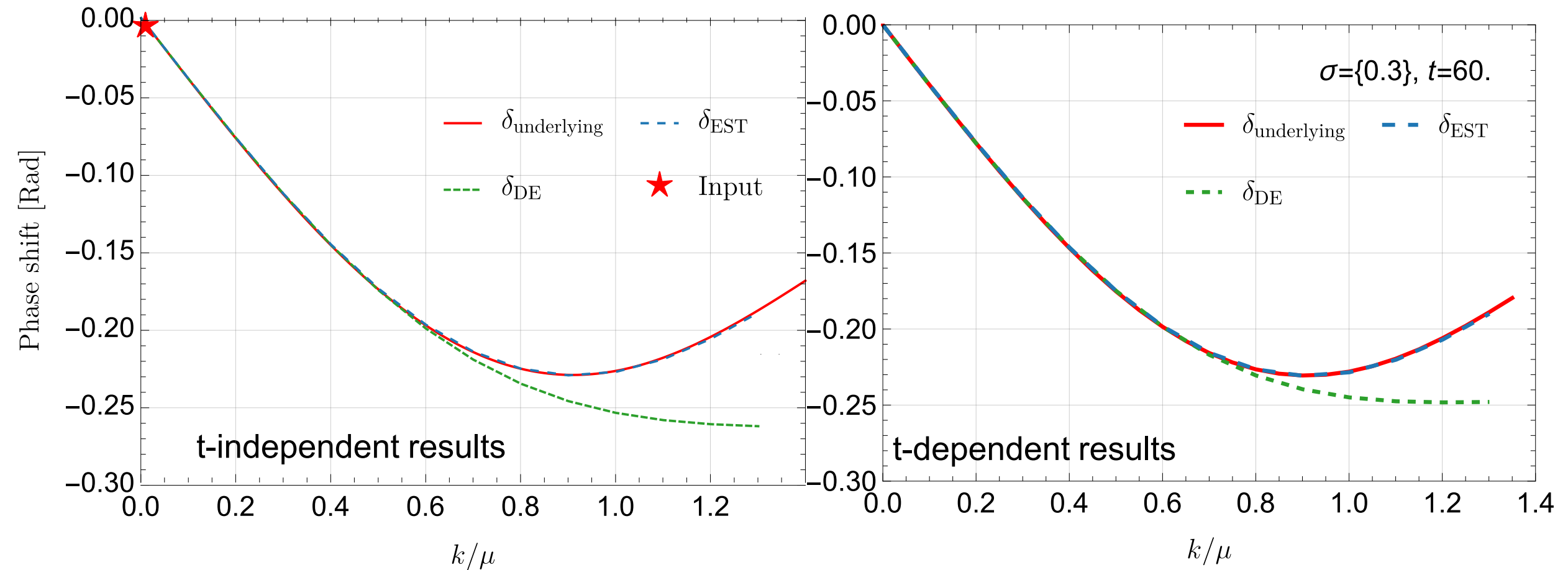
$$\tilde{R}(t=0, x) = \frac{\sigma^2 e^{-\sigma x}}{4\pi}$$

- ▶ Evaluate t=60
- ▶ Two $\sigma = \{0.3, 0.6\}$ as two inputs

Separatable interaction

- The EST methods give the accurate potential in LO
- The DE method is convergent

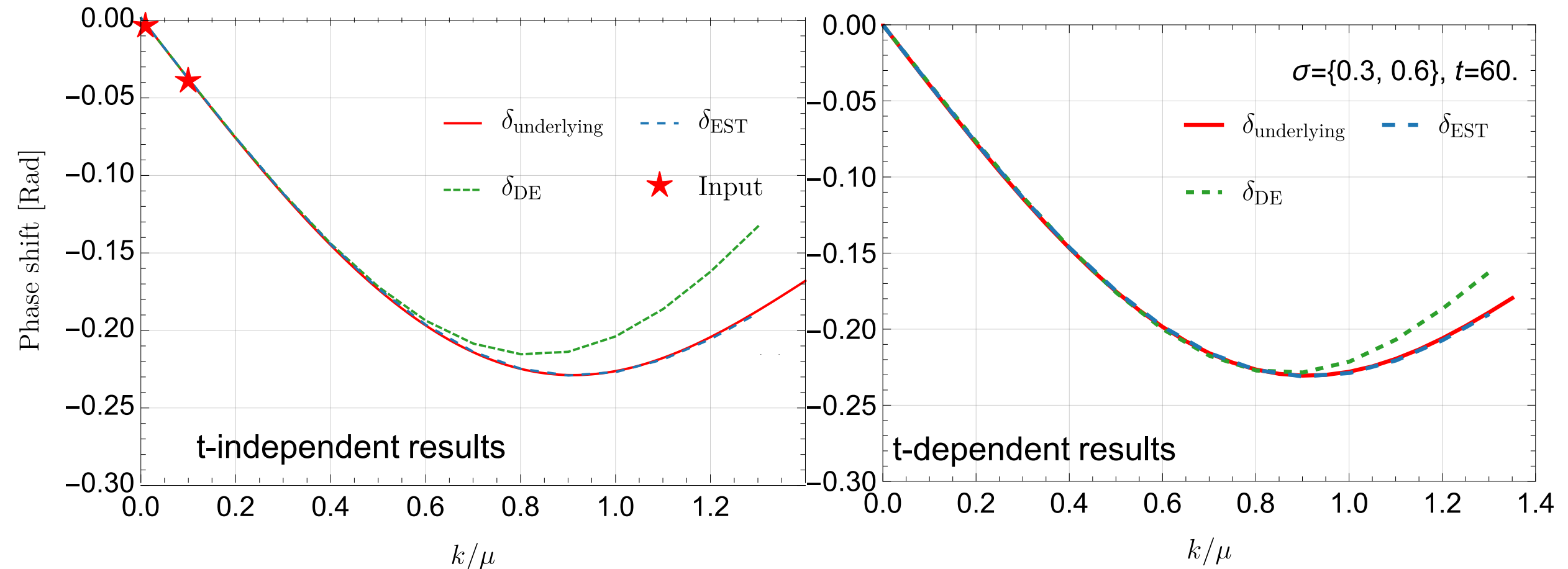
$$V(\mathbf{r}, \mathbf{r}') = \omega \frac{e^{-\mu r}}{r} \frac{e^{-\mu r'}}{r'}$$



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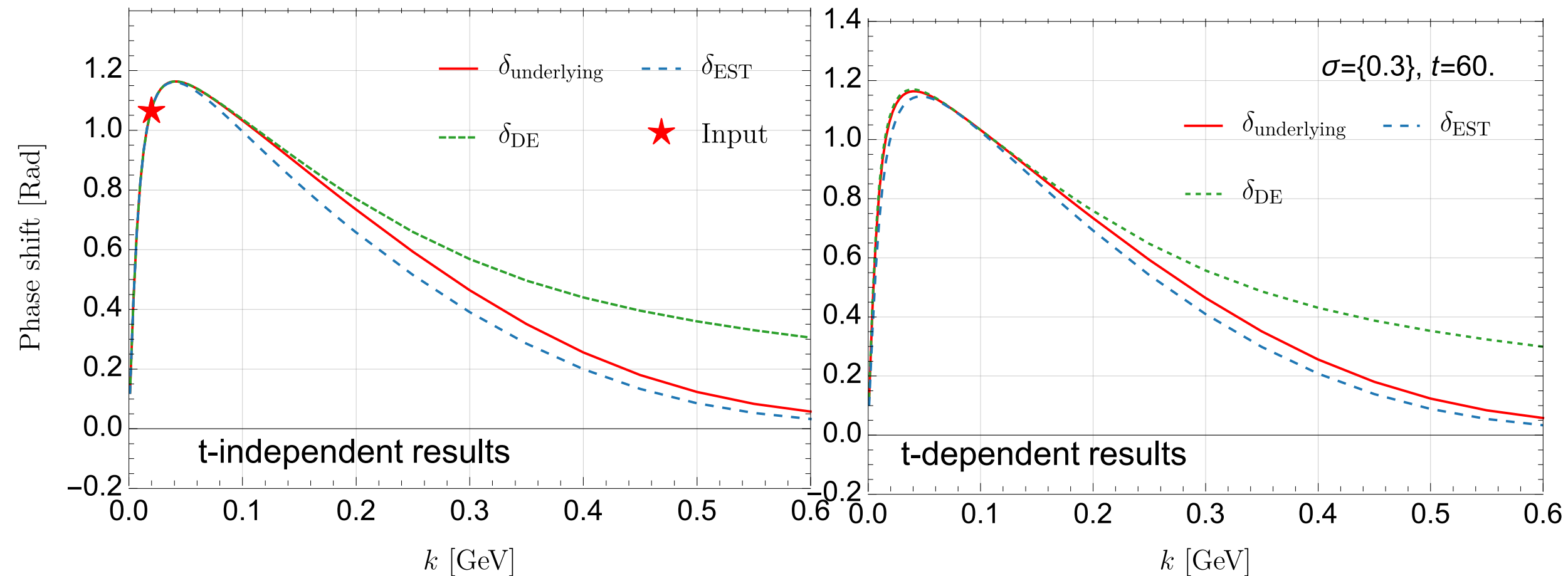


Physical interaction

- Including both separatable part and local part
- The performance of EST method is better
- In t-dependent methods, singular potential

$$V_{ctc}(\mathbf{p}, \mathbf{p}') = C e^{-\frac{p^2 + p'^2}{\Lambda^2}},$$

$$V_{ope}(\mathbf{q}) = -\frac{g_A}{4F_\pi^2} \left(\frac{\boldsymbol{\sigma}_1 \cdot \mathbf{q} \boldsymbol{\sigma}_2 \cdot \mathbf{q}}{q^2 + m_\pi^2} + C_{sub} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \right) e^{-\frac{q^2 + m_\pi^2}{\Lambda^2}}$$

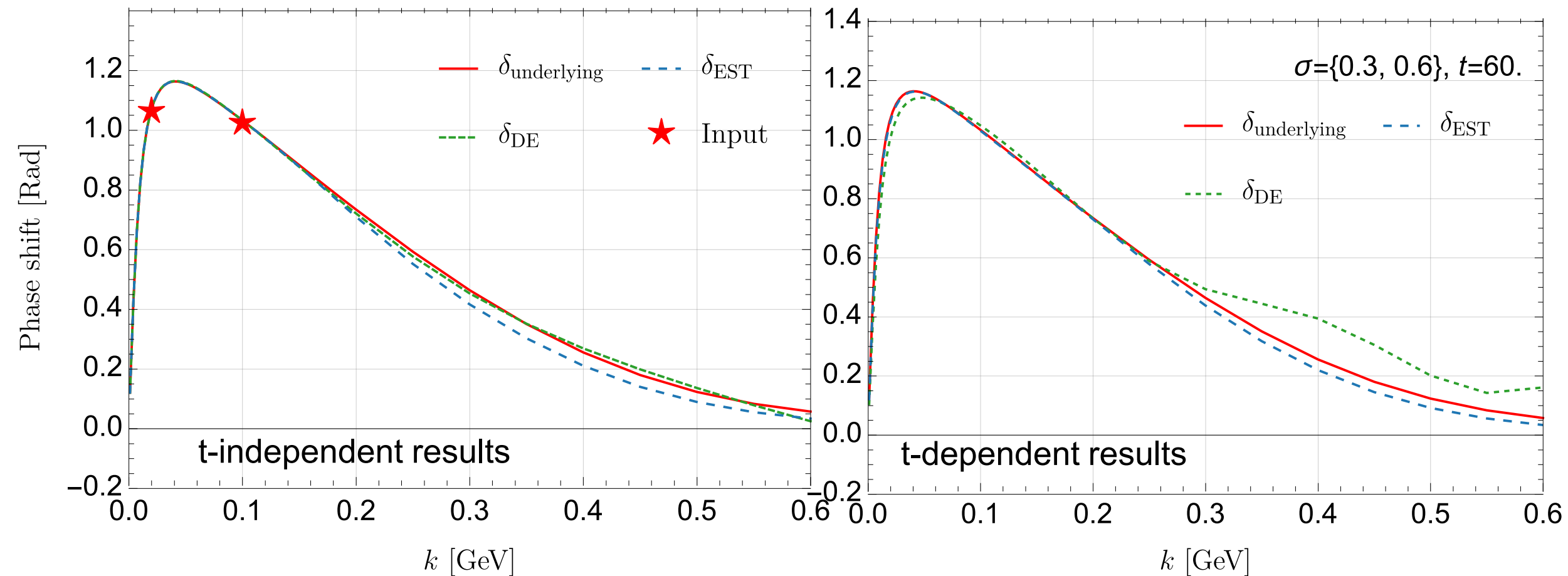


Physical interaction

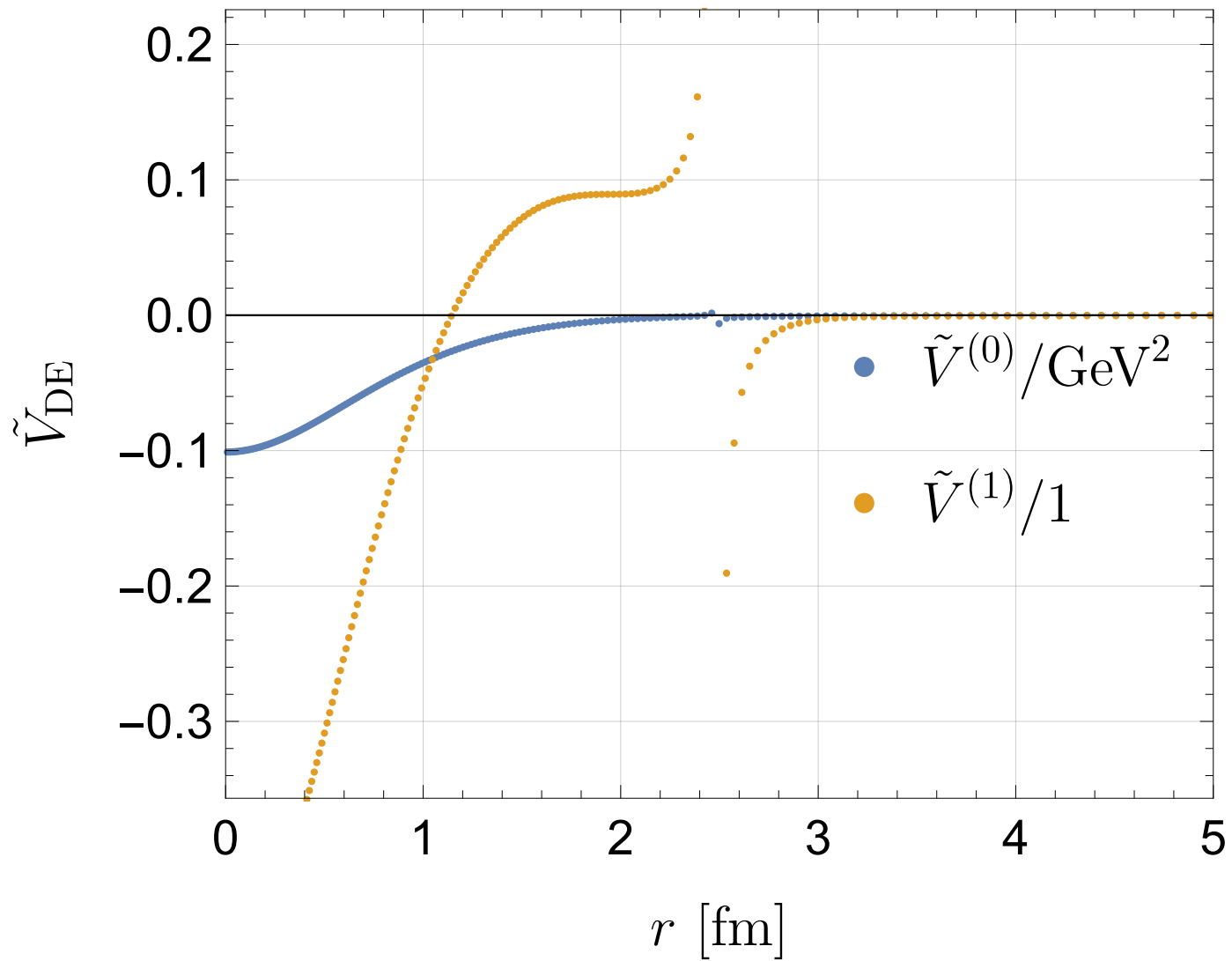
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Singularity in potential

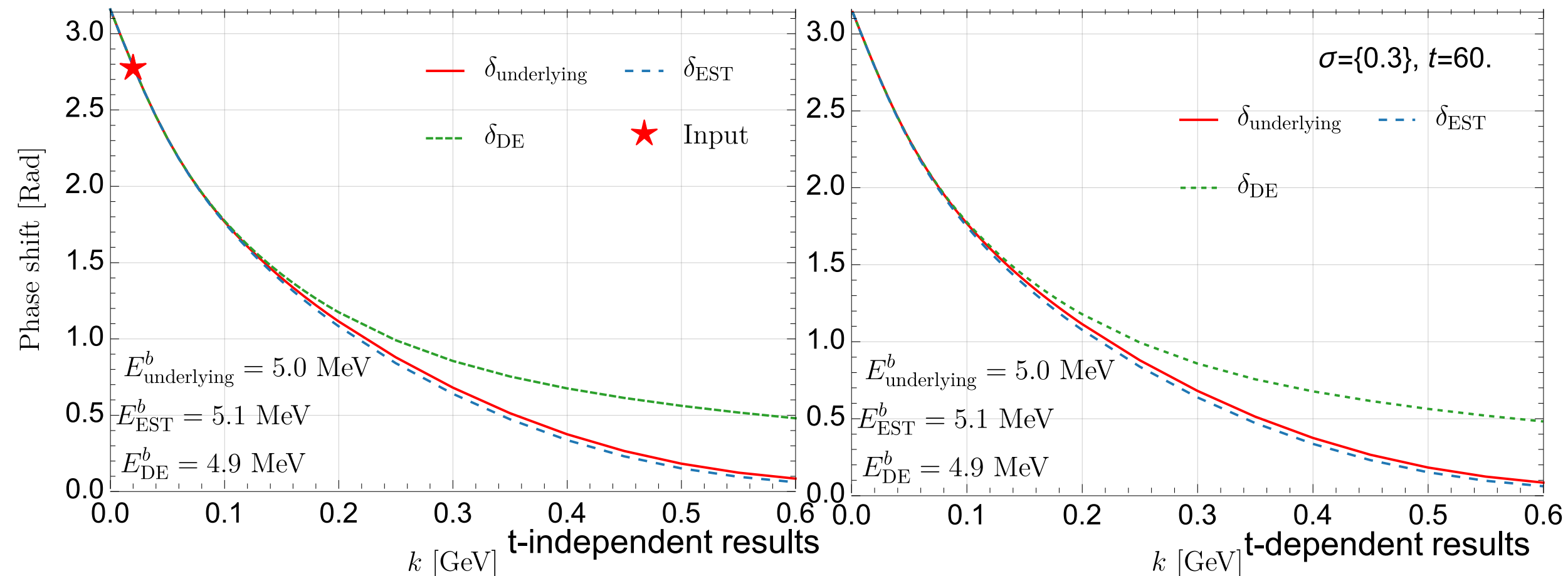


Bound state

- At LO, both EST and DE method give reasonable binding energy
- The EST method perform better in phase shift
- Singular potential in DE at NLO

$$V_{ctc}(\mathbf{p}, \mathbf{p}') = C e^{-\frac{p^2+p'^2}{\Lambda^2}},$$

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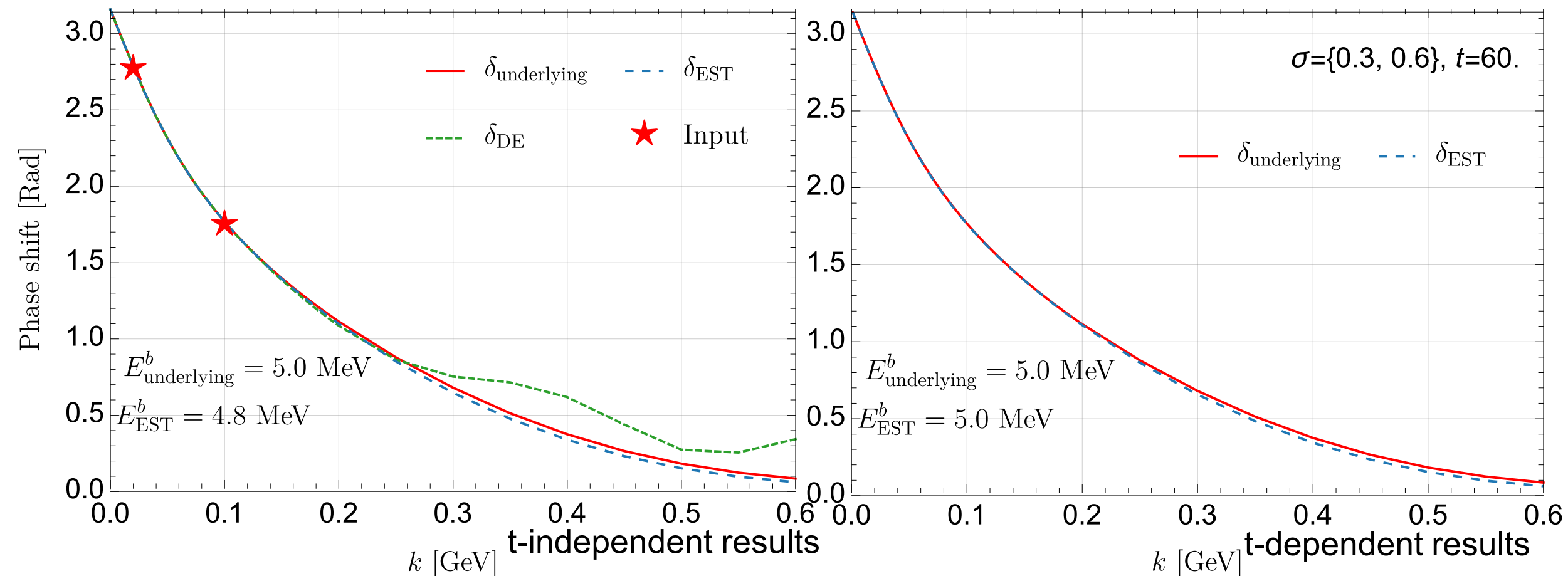


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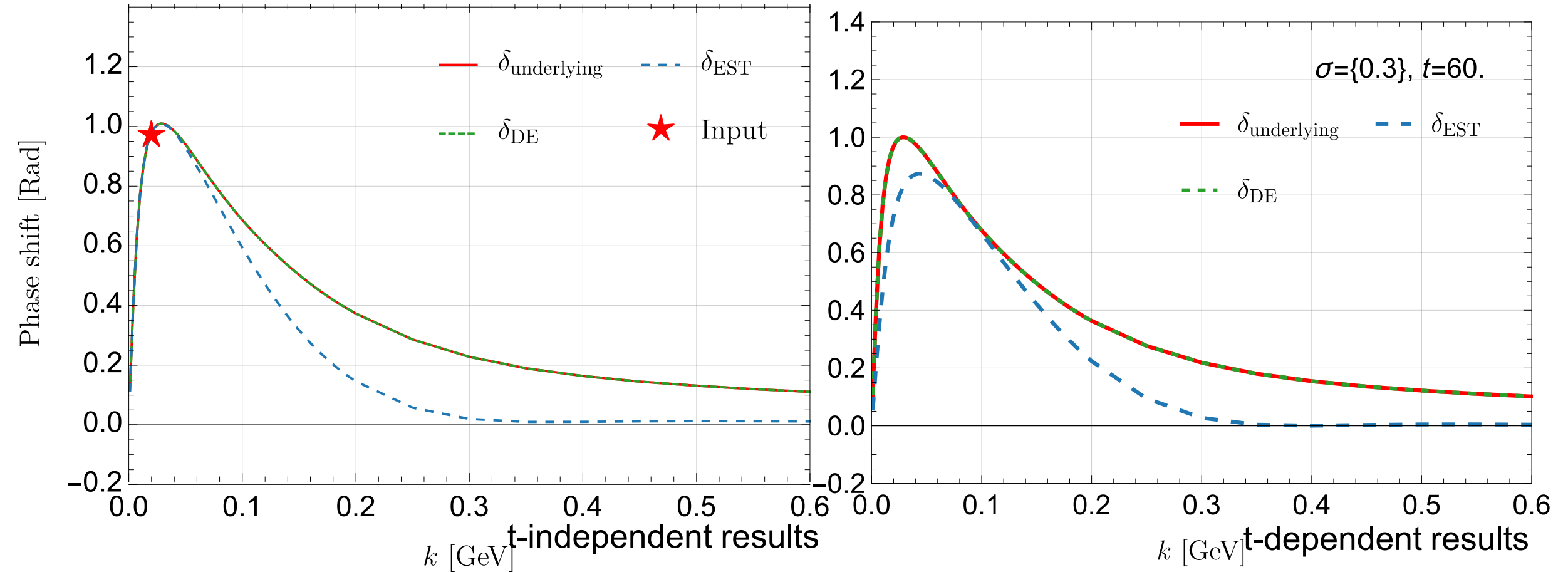


Local interaction

- The DE method gives the accurate results at LO
- Convergent EST results, not bad performance

$$V_{ctc}(\mathbf{p}, \mathbf{p}') = C e^{-\frac{p^2 + p'^2}{\Lambda^2}},$$

$$V_{ope}(\mathbf{q}) = -\frac{g_A}{4F_\pi^2} \left(\frac{\boldsymbol{\sigma}_1 \cdot \mathbf{q} \boldsymbol{\sigma}_2 \cdot \mathbf{q}}{q^2 + m_\pi^2} + C_{sub} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \right) e^{-\frac{q^2 + m_\pi^2}{\Lambda^2}}$$

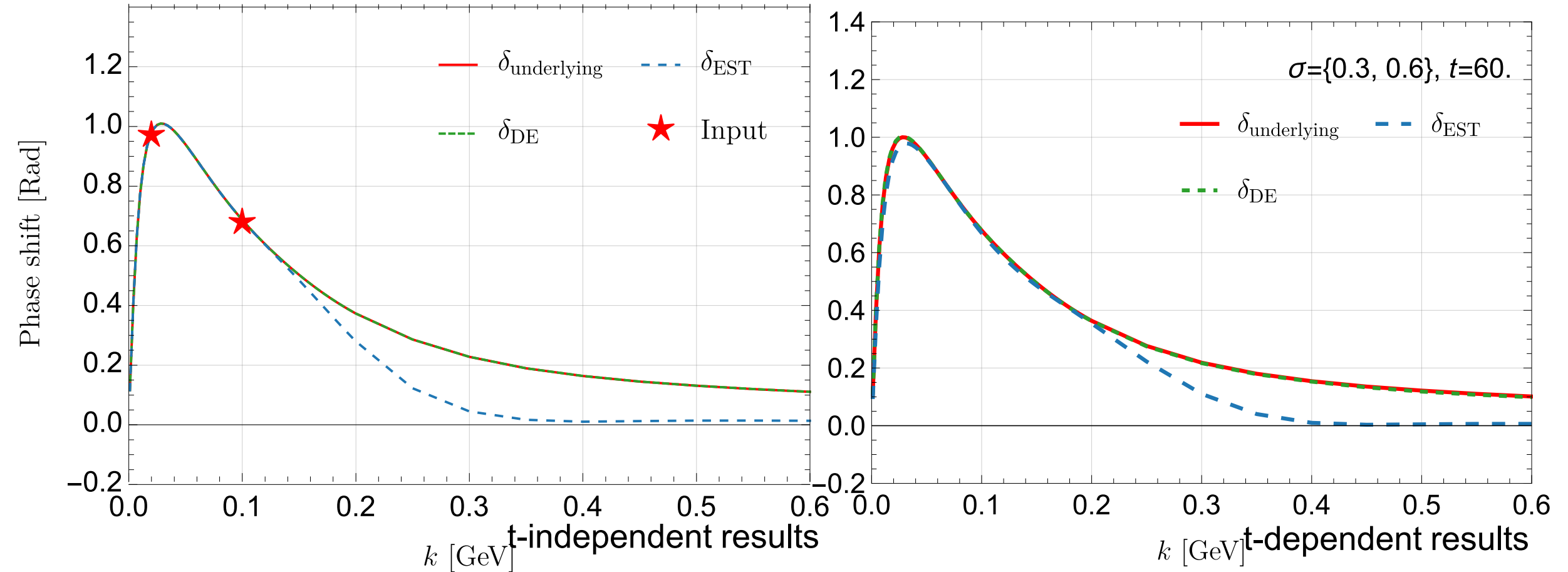


Local interaction

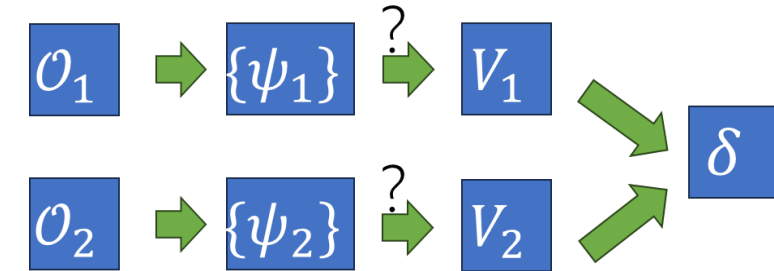
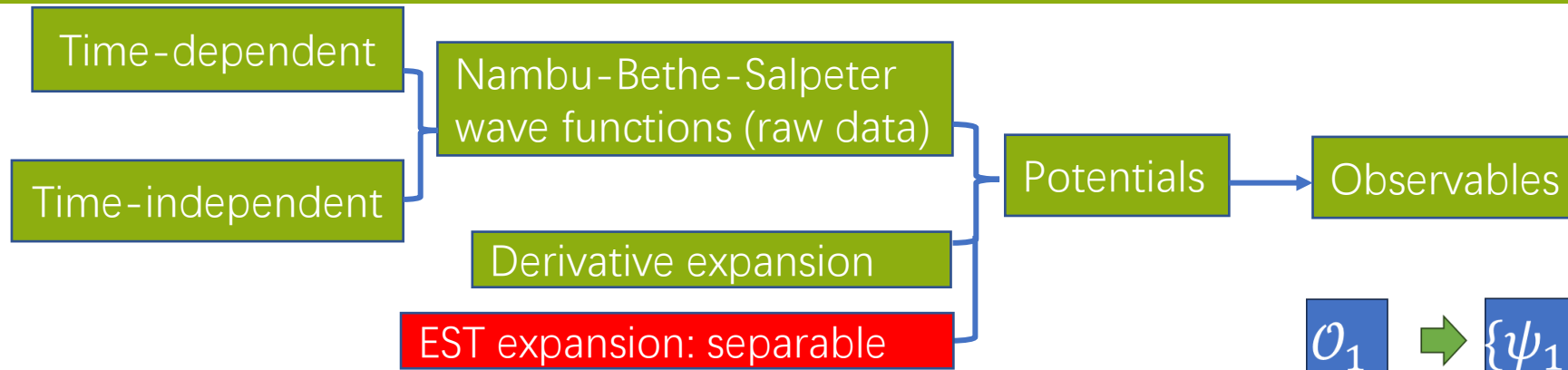
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Summary



- Re-emphasize some concepts
 - ▶ Potential and non-asymptotic wave function are not observable
 - ▶ the interpolating operators \Rightarrow HAL QCD
 - ▶ A small number of wave functions can NOT determine the potential definitely
 - ▶ One cannot rule out the nonlocal potential either in principle or phenomenologically
- Derivative expansion VS EST expansion
 - ▶ For local potential, DE performs better, EST is not so bad (converge)
 - ▶ For separable potential EST perform better
 - ▶ For LO chiral nuclear force, EST perform better
- EST provide a alternative way to extract potential
 - ▶ Avoid possible singular potential
 - ▶ Changing potential representation takes less pains than changing operators to re-simulate
 - ▶ A way to estimate the systemic uncertainty
 - ▶ Combing EST and DE: short-range: EST, long-range: DE

Thanks for your attention!

Back up

TABLE 1 | Summary of binding energies [MeV] for $NN(^1S_0)$, $NN(^3S_1)$, and H -dibaryon in lattice QCD.

Collaboration	References	N_f	m_π	$-\Delta E(^1S_0)$	$-\Delta E(^3S_1)$	$-\Delta E(H)$
The direct method						
YKU2011	[35]	0	800	4.4 (1.2)	7.5 (1.0)	—
YIKU2012	[36]	2+1	510	7.4 (1.4)	11.5 (1.3)	—
NPL2015	[37]	2+1	450	12.5 ($^{+3.0}_{-5.0}$)	14.4 ($^{+3.2}_{-2.6}$)	—
NPL2012	[38]	2+1	390	7.1 (9.0)	11 (13)	13.2 (4.4)
YIKU2015	[39]	2+1	300	8.5 ($^{+1.7}_{-0.9}$)	14.5 ($^{+2.5}_{-1.1}$)	—
NPL2013	[40]	3	810	15.9 (3.8)	19.5 (4.8)	74.6 (4.7)
NPL2017	[41]	3	810	20.6 ($^{+3.3}_{-2.9}$)	27.9 ($^{+3.8}_{-2.7}$)	—
CalLat2017	[42]	3	810	21.8 ($^{+3.3}_{-5.8}$)	30.7 ($^{+2.5}_{-3.0}$)	—
		3		8.35 (1.1)*	3.3 ($^{+1.2}_{-0.9}$)	—
Mainz2018	[43]	3 [†]	960	0	—	19 (10)
		2+1 [†]	440	—	—	18.8 (5.5)*
The HAL QCD method						
IAH2007	[26]	0	530	0	0	—
AHI2009	[23]	0	380, 530, 730	0	0	—
HAL2012	[44]	3	1171	0	0	49.1(6.5)
		3	1015	0	0	37.2(4.4)
		3	837	0	0	37.8(5.2)
		3	672	0	0	33.6(5.9)
		3	469	0	0	26.0(6.5)
HAL2012a	[34]	2+1	701	0	—	—
HAL2013	[45]	2+1	411, 570, 701	0	—	—

NPL2013, NPL2017, and CalLat2017 employed the same set of gauge configurations. CalLat2017 found two states in each channel. In Mainz2018, dynamical 2-flavor with quenched strange quark configurations are employed and N_f in the table (with [†] symbol) denotes the information in the valence quark sector. All values of ΔE correspond to those in the infinite volume limit except ones with *, which are values on the finite volumes. The number 0 in ΔE indicates the system is unbound in this channel.

the BS wave function in the “outer region” ($r > R$) satisfies the Helmholtz equation, $((W/2)^2 - \nabla^2 + m_N^2)\psi_{\alpha\beta}(\mathbf{r}) = -(\nabla^2 + k^2)\psi_{\alpha\beta}(\mathbf{r}) = 0$, up to an exponentially small correction. Here the “asymptotic momentum” k is related to the total energy W through the relation, $W = 2\sqrt{k^2 + m_N^2}$. To make a formal resemblance with the non-relativistic case, we introduce the “effective center of mass energy”, $E = k^2/(2\mu) = k^2/m_N$.¹⁵⁾ As shown in Appendix A, using the unitarity of the

Separable representation

● The problem: $V|R^{(i)}\rangle = |K^{(i)}\rangle$

● Separable representation I:

$$V = \sum_{ij} C_{ij} |K^{(i)}\rangle \langle R^{(j)}|, \quad C_{im} \langle R^{(m)} | R^{(j)} \rangle = \delta_{ij}$$

Aoki:2009ji

▶ Bad performance

▶ In the outer region: $K^{(i)}(r) = (\frac{d^2}{dr^2} + k_i^2)\psi_{k_i}(r) = 0$, $R^{(i)}(r) = \psi_{k_i}(r) \neq 0$

● Separable representation II, Ernst-Shakin-Thaler (EST) method

$$V = \sum_{mn} |K^{(m)}\rangle \Lambda_{mn} \langle K^{(n)}|, \quad \Lambda_{mn} \langle K^{(n)} | R^{(i)} \rangle = \delta_{mi}$$

▶ In the outer region: $K^{(i)}(r) = 0$

Ernst:1973zzb,Haidenbauer:1984dz

▶ Application: on-shell and off-shell equivalent separable potentials of NN Paris potentials

$$\left[\sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} \right] \phi(p) + \frac{1}{(2\pi)^3} \int U(p, p') p'^2 \phi(p') = E\phi(p)$$

$$\left[m_1 + m_2 + \frac{q^2}{2m_1} + \frac{q^2}{2m_2} \right] \psi(q) + \frac{1}{(2\pi)^3} \int V(q, p') p'^2 \psi(p') = E\psi(q)$$

$$\sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} = m_1 + m_2 + \frac{q^2}{2m_1} + \frac{q^2}{2m_2}$$

$$p^2 dp = h(q)^2 q^2 dq$$

$$|h(q)|^2 = \frac{p}{q} \frac{\omega_{12}}{\mu_{12}}$$

$$\left[m_1 + m_2 + \frac{q^2}{2m_1} + \frac{q^2}{2m_2} \right] \phi(p(q)) + \frac{1}{(2\pi)^3} \int U(p(q), p'(q)) h(q')^2 q'^2 \phi(p'(q')) = E\phi(p(q))$$

$$\left[m_1 + m_2 + \frac{q^2}{2m_1} + \frac{q^2}{2m_2} \right] h(q)\phi(p(q)) + \frac{1}{(2\pi)^3} h(q) \int U(p(q), p'(q')) h(q')^2 q'^2 \phi(p'(q')) = Eh(q)\phi(p(q))$$

if $\psi(q) = h(q)\phi(p(q))$ and $h(q)U(p(q), p'(q'))h(q') = V(q, q')$

we can get

$$\left[m_1 + m_2 + \frac{q^2}{2m_1} + \frac{q^2}{2m_2} \right] \psi(q) + \frac{1}{(2\pi)^3} \int V(q, p') p'^2 \psi(p') = E\psi(q)$$

Similarity renormalization group

$$H_s = U(s) H U^\dagger(s) \equiv T_{\text{rel}} + V_s, \quad (1)$$

$$\frac{dU(s)}{ds} = [\eta(s), H_s], \quad (2)$$

with

$$\eta(s) = \frac{dU(s)}{ds} U^\dagger(s) = -\eta^\dagger(s). \quad (3)$$

Choosing $\eta(s)$ specifies the transformation. Here we make perhaps the simplest choice [10],

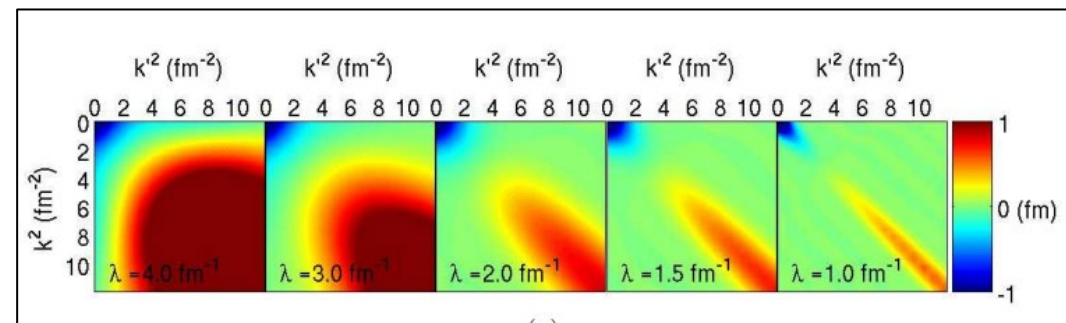
$$\eta(s) = [T_{\text{rel}}, H_s], \quad (4)$$

which gives the flow equation,

$$\frac{dH_s}{ds} = [[T_{\text{rel}}, H_s], H_s]. \quad (5)$$

Other choices will be studied elsewhere [14].

$$\begin{aligned} \frac{dV_s(k, k')}{ds} = & -(k^2 - k'^2)^2 V_s(k, k') \\ & + \frac{2}{\pi} \int_0^\infty q^2 dq (k^2 + k'^2 - 2q^2) \\ & \times V_s(k, q) V_s(q, k'). \end{aligned} \quad (6)$$



SRG evolution

- Soften the short range interaction
- Improve the convergence of the many-body problem

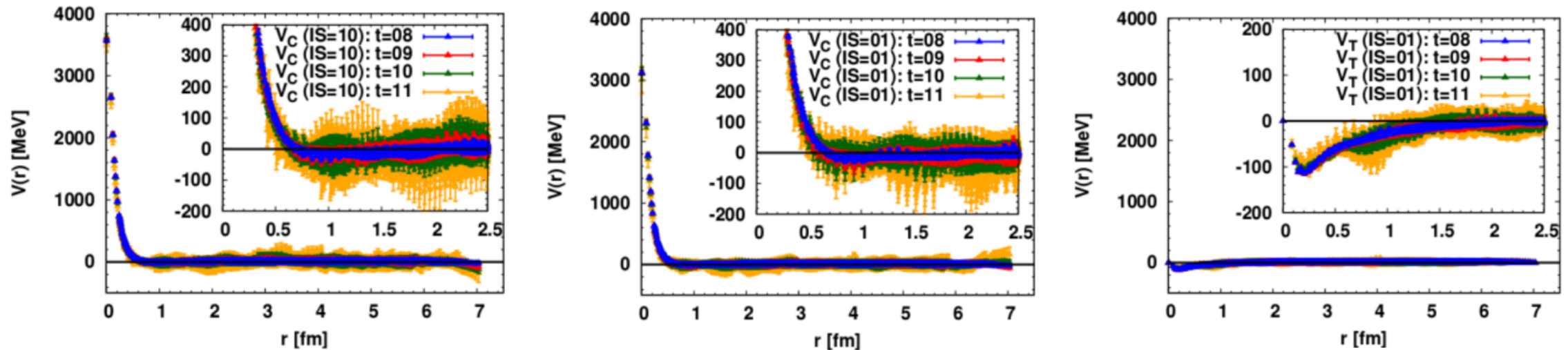
Highlight of HALQCD results: NN interaction

- lattice Setting: $m_\pi = 146$ MeV, $m_K = 525$ MeV, $a = 0.0846$ fm, $L = 8.1$ fm, 96^4

⇒ Almost physical pion mass, very large box size, the finite volume effect is neglected

⇒ NN: 1S_0 central potential, 3S_1 central potential, $^3S_1 - ^3D_1$ tensor potential

Doi:2017zov



D^*D interaction: 2 pion tails

- lattice Setting: $m_\pi = 146$ MeV, $m_K = 525$ MeV, $a = 0.0846$ fm, $L = 8.1$ fm, 96^4
- D^*D , ϕN interaction...

Lyu:2022imf,Lyu:2023xro

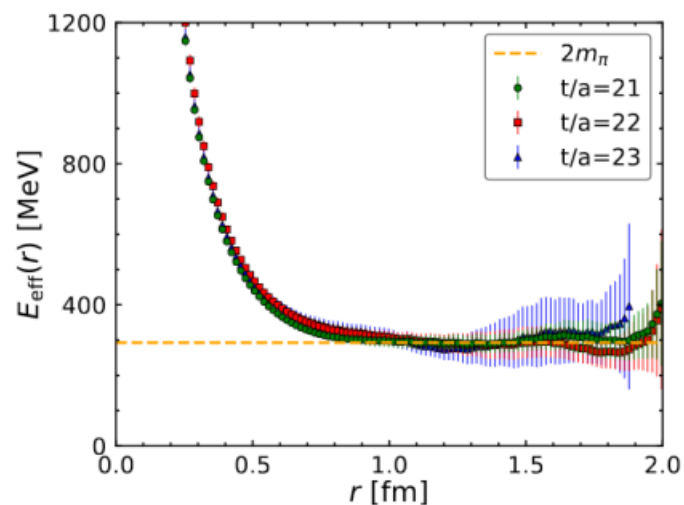
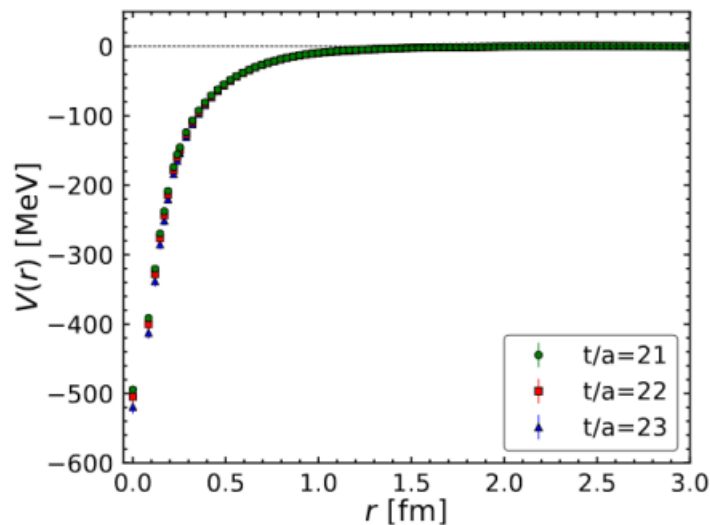


FIG. 2. The D^*D potential $V(r)$ in the $I = 0$ and S -wave channel at Euclidean time $t/a = 21$ (green circles), 22 (red squares), and 23 (blue triangles).

Fitting the potential with:

$$V_{fit}(r) = \sum_{i=1,2} a_i e^{-r^2/b_i^2} + a_3 \frac{e^{-2mr}}{r^2} \quad (65)$$

once a_3 is determined

define

$$E_{eff}(r) = -\frac{\ln[-V(r)r^2/a_3]}{r} \quad (66)$$

one get a plateau at $E_{eff} = 2m$

- No one-pion exchange interaction: $\frac{1}{u} = 4.1$ fm

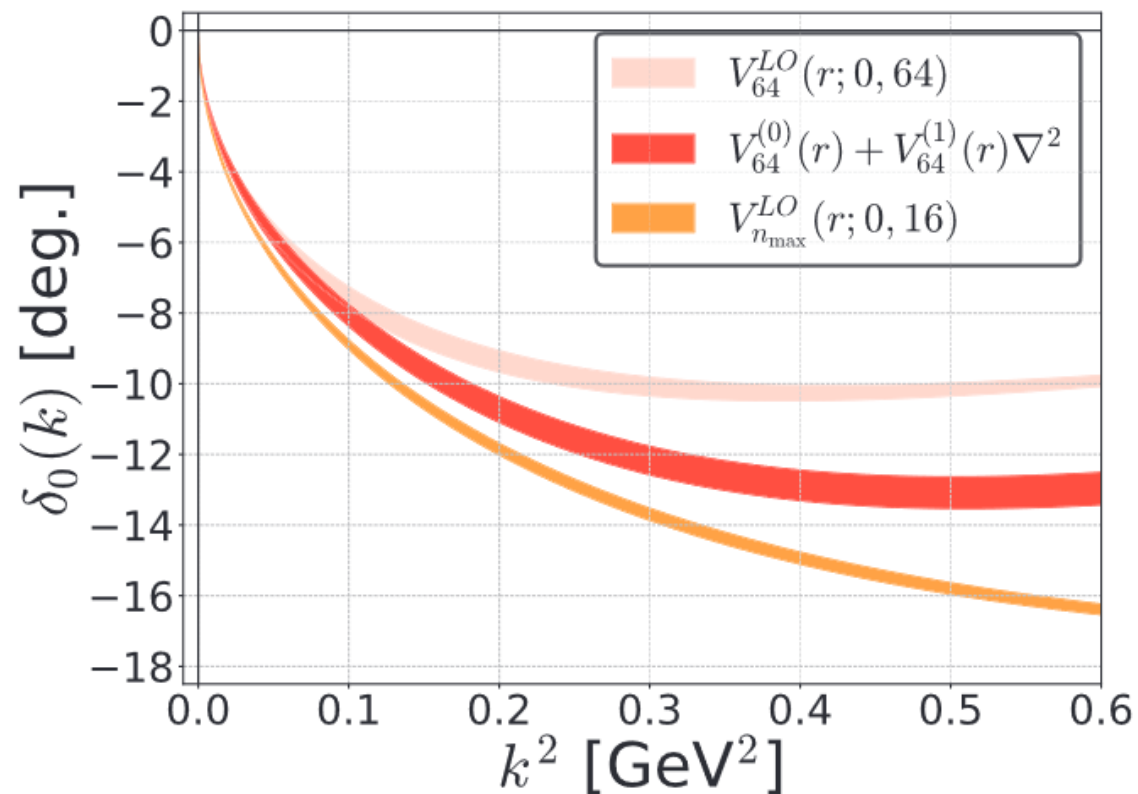


Fig. 5. The phase shifts of the S-wave $I = 2$ $\pi\pi$ scattering from the potential in the point-sink scheme (LO: orange) and the smeared-sink scheme (LO: pink, NLO: red) as a function of k^2 .

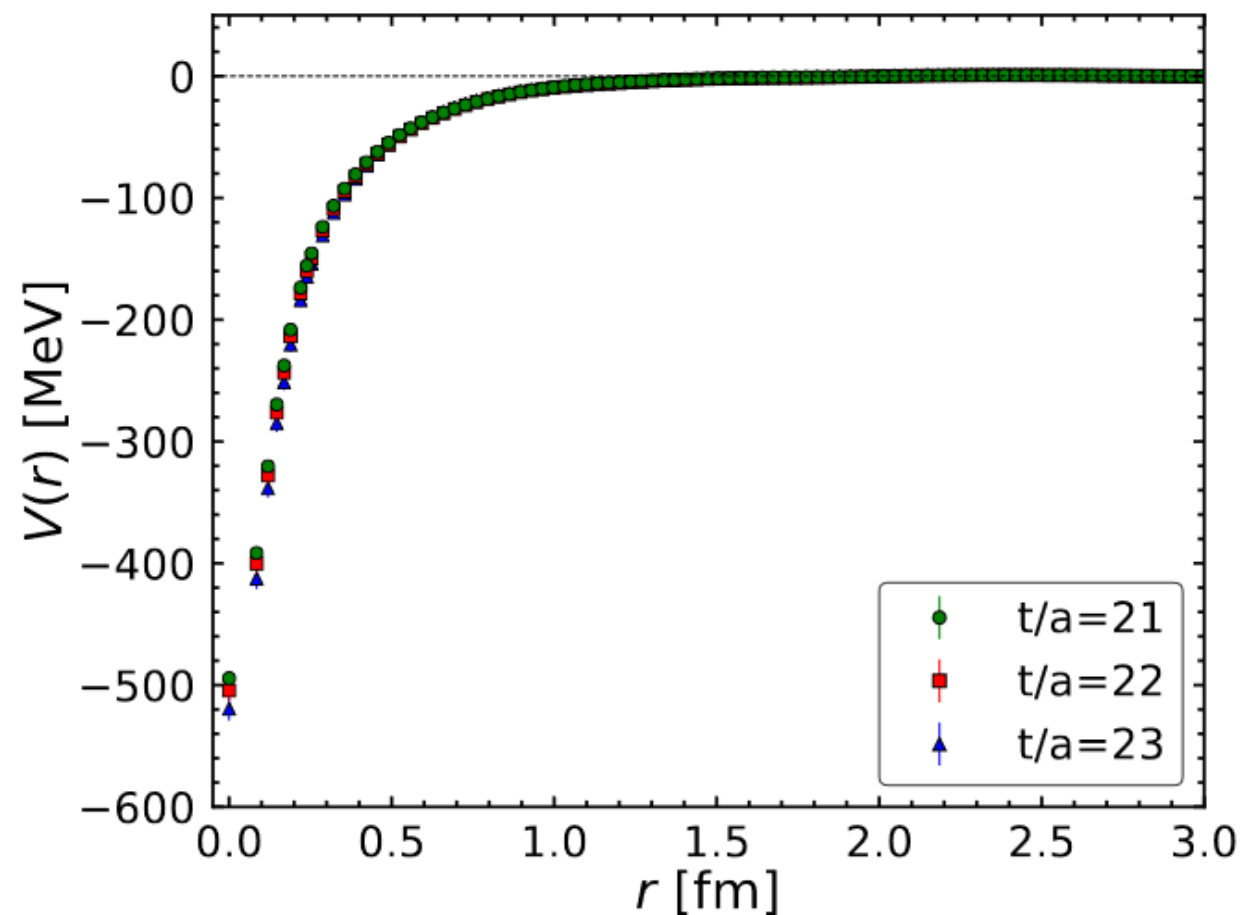
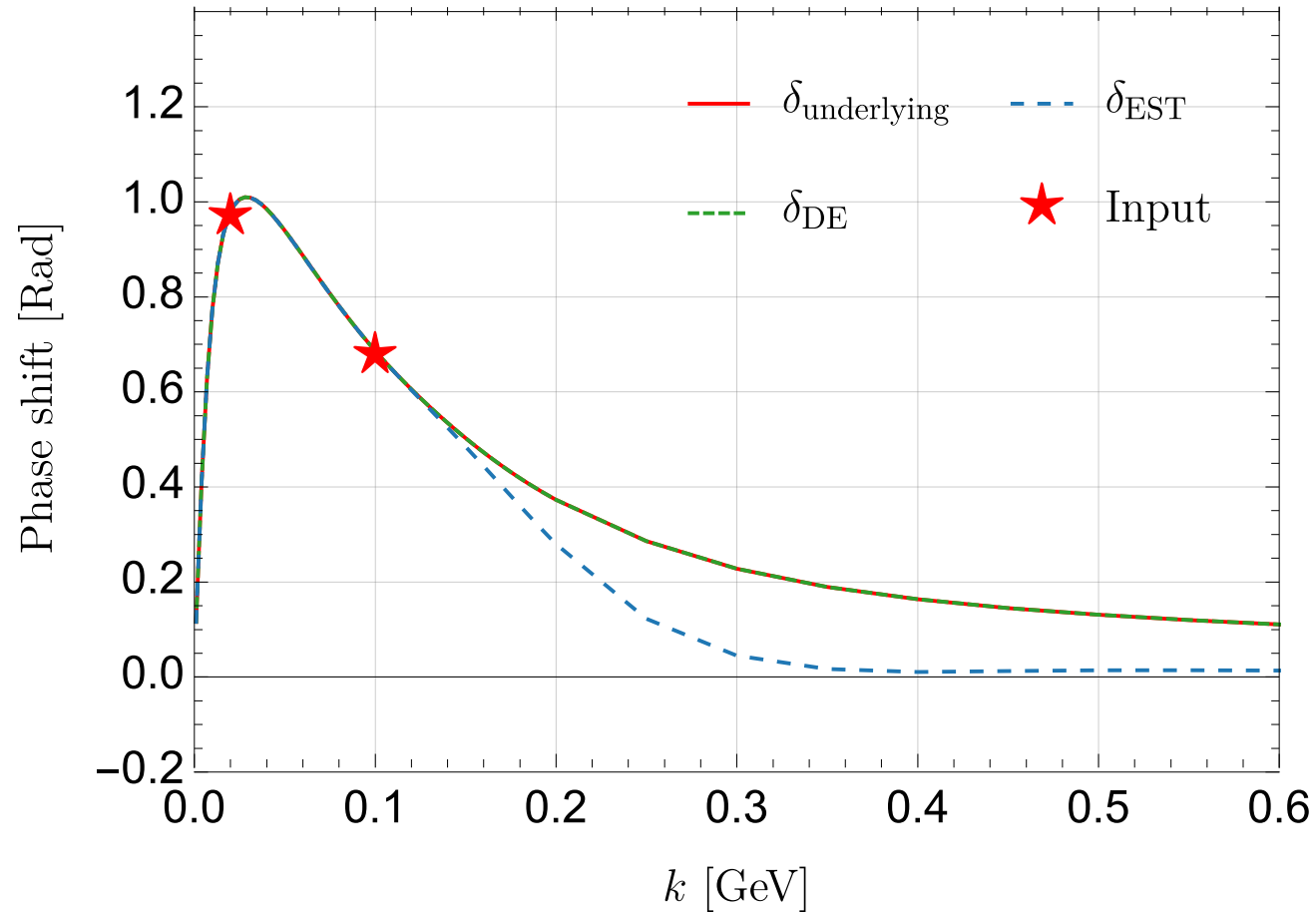


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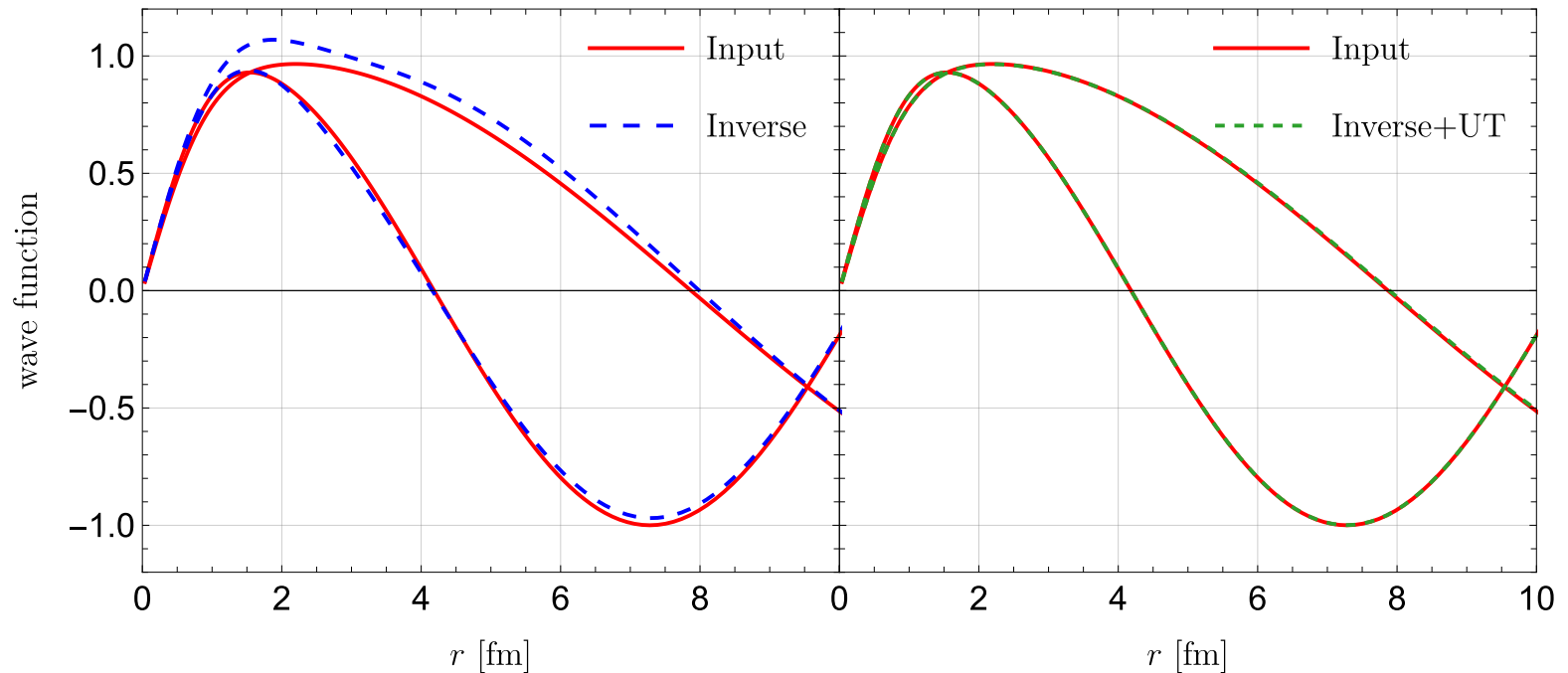
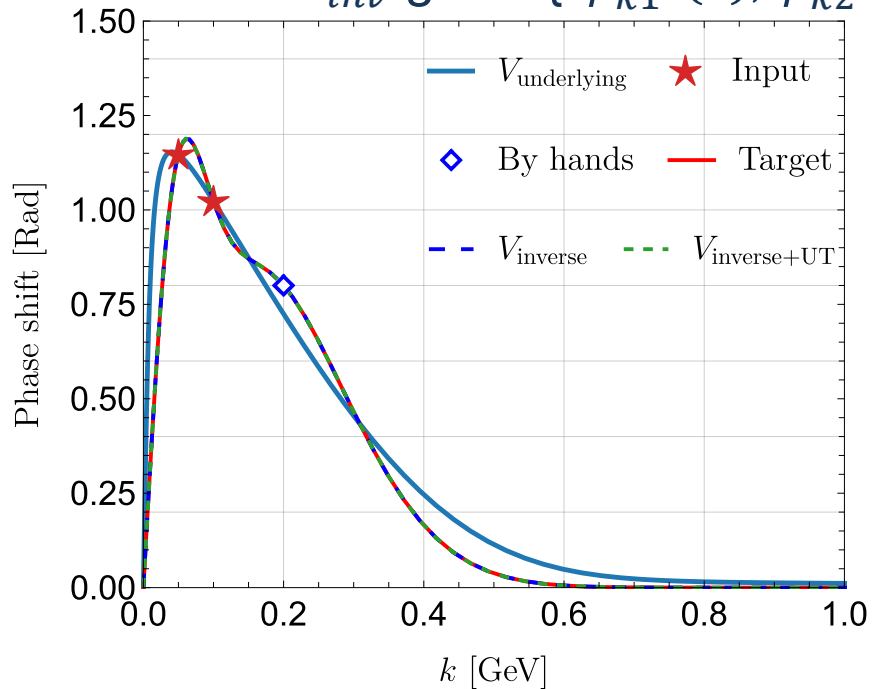
- Extracting potential from NBS is not a expansion of small quantities
- It is more like a interpolating and extrapolating
- Self-consistence test also make sense



A small number of wave functions

1. Underlying potential $V_{underlying}$ give its phase shift $\delta(k)$
2. Using two wave functions as input $\{\psi_{k_1}(r), \psi_{k_2}(r)\}$ with phase shifts $\{\delta(k_1), \delta(k_2)\}$
3. Find a $\delta_{tar}(k)$ go thorough $\{\delta(k_1), \delta(k_2)\}$ and the third phase shift $\delta_{by-hand}(k_3)$ assigned by hand
4. Find a potential $V_{inverse}$ permit $\delta_{tar}(k)$
 - ▶ many choices: i.e. a separable potential
5. The V_{inv} gives $\{\psi_{k_1}^{inv}(r), \psi_{k_2}^{inv}(r)\}$ different with $\{\psi_{k_1}(r), \psi_{k_2}(r)\}$

Tabakin:1969mr



A small number of wave functions

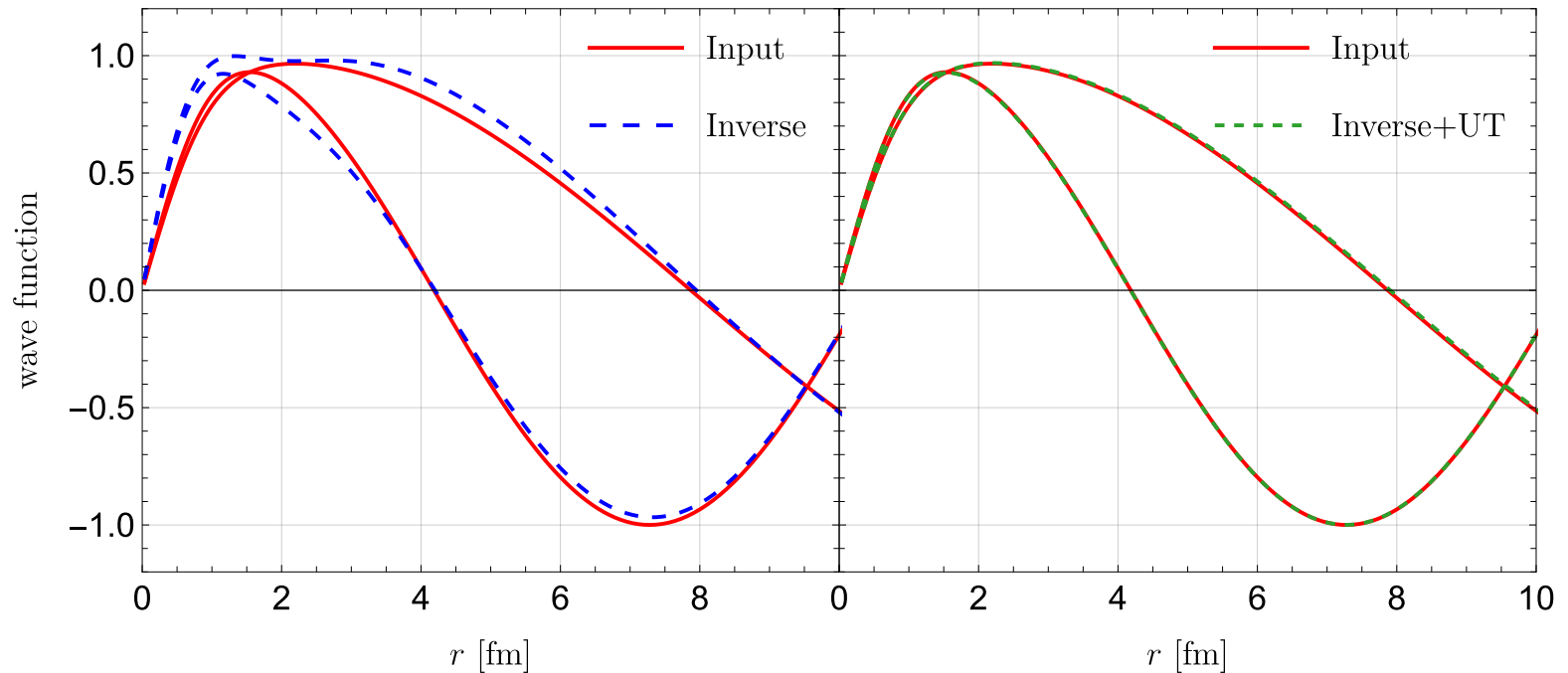
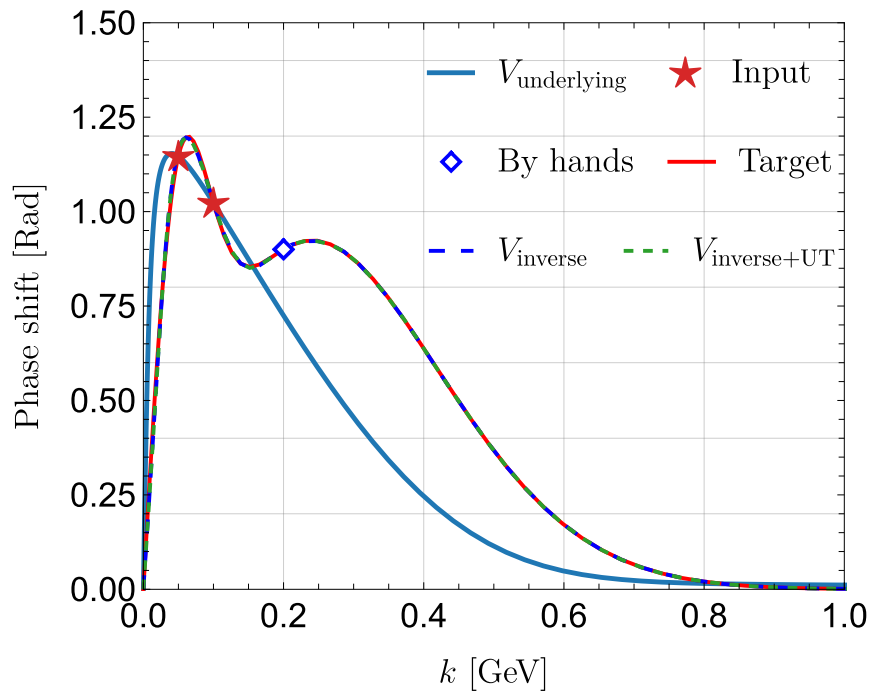
6. Construct an unitary trans. (UT): $U|\psi_{k_i}^{inv}\rangle = |\psi_{k_i}\rangle$

Ernst:1973utx

$$|f_i\rangle = |\psi_{k_i}\rangle - |\psi_{k_i}^{inv}\rangle, \quad U - 1 \equiv \sum_{mn} |f_m\rangle \Lambda_{mn} \langle f_n|, \quad \Lambda_{mn} \langle f_n | \psi_i \rangle = \delta_{mi}$$

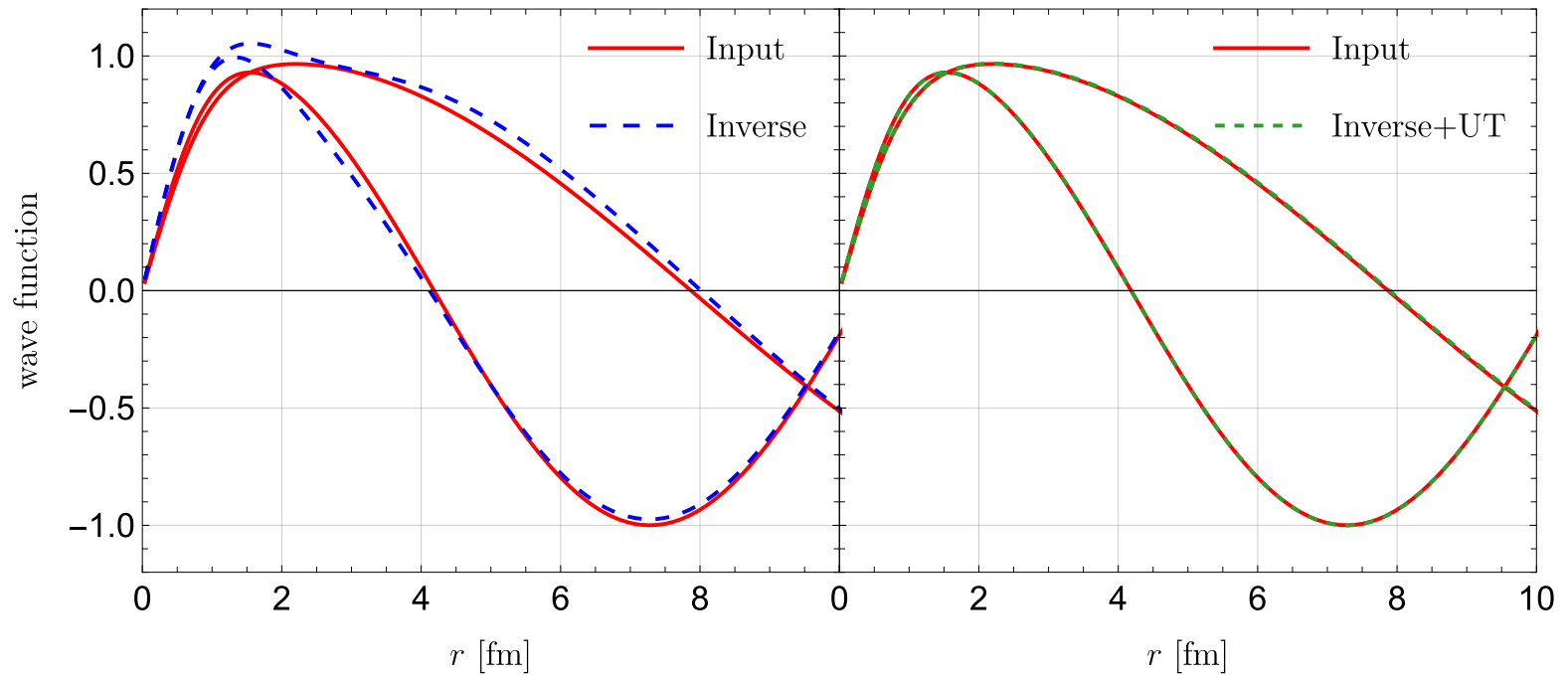
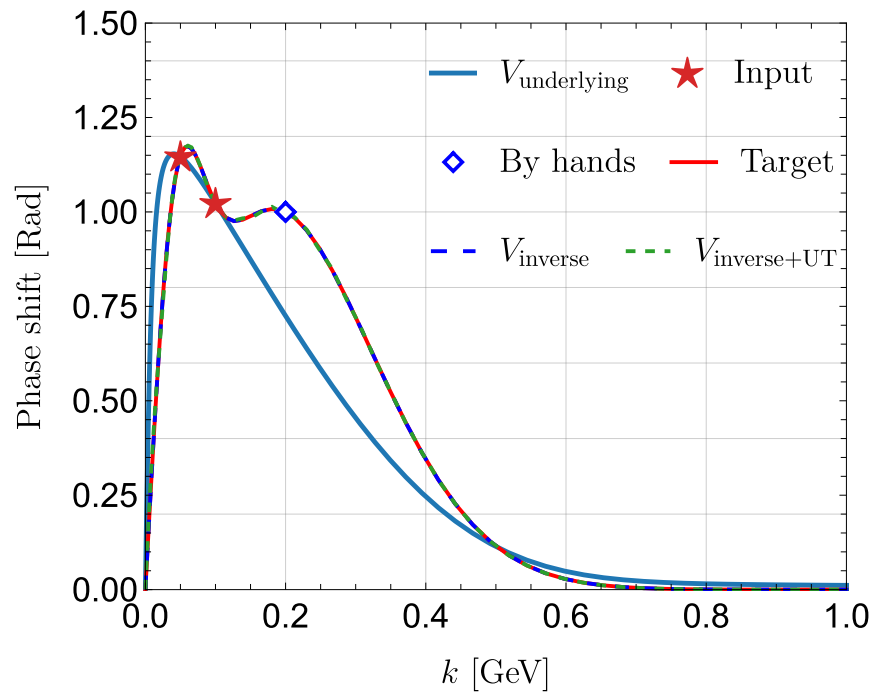
7. V^{inv+UT} permit the $\{\psi_{k_1}(r), \psi_{k_2}(r)\}$

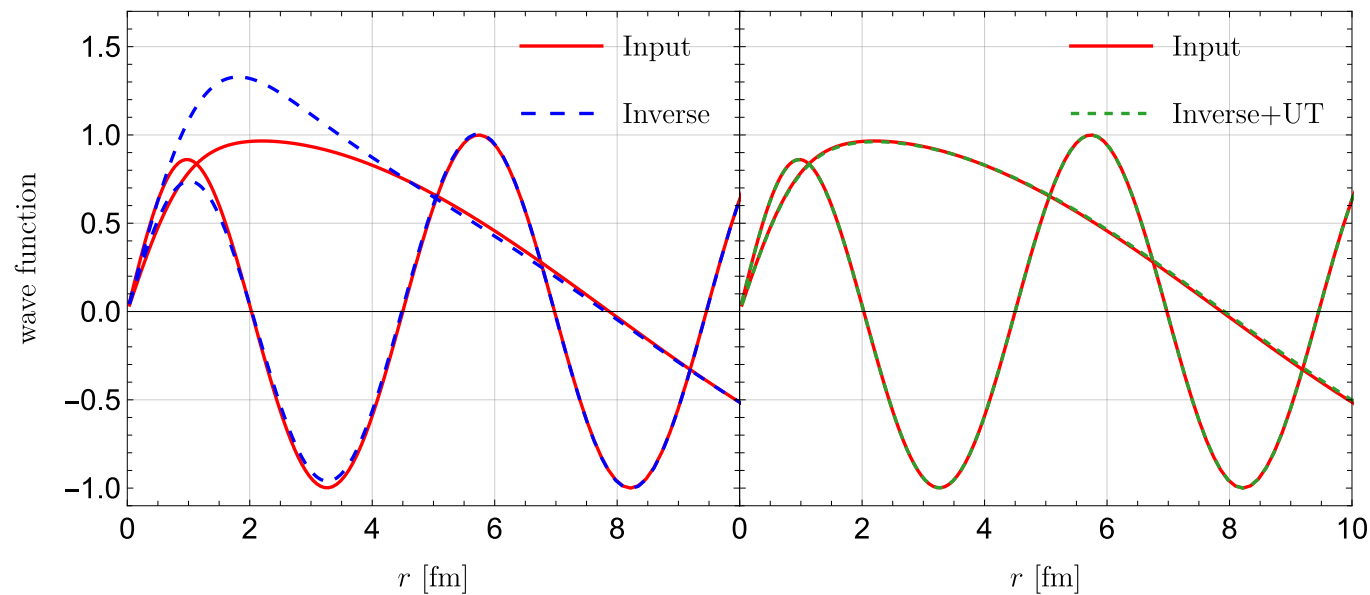
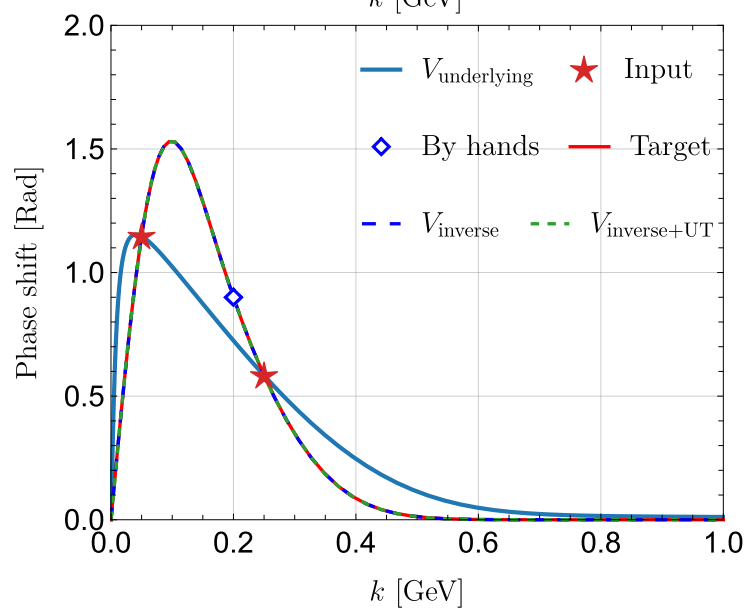
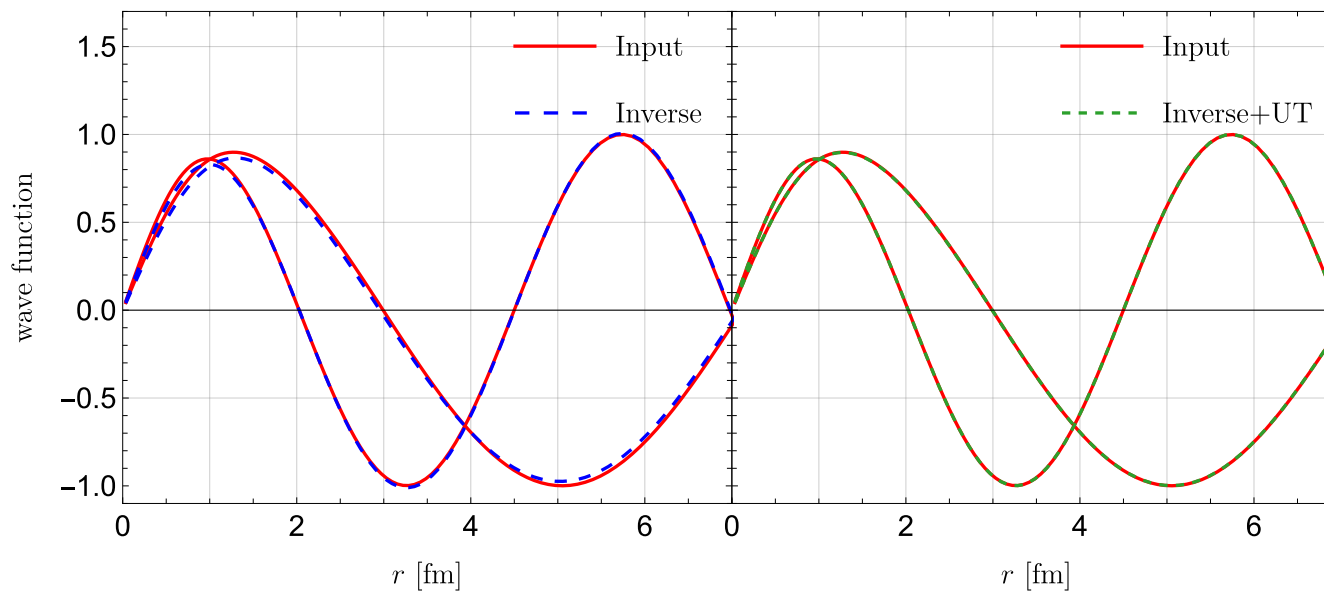
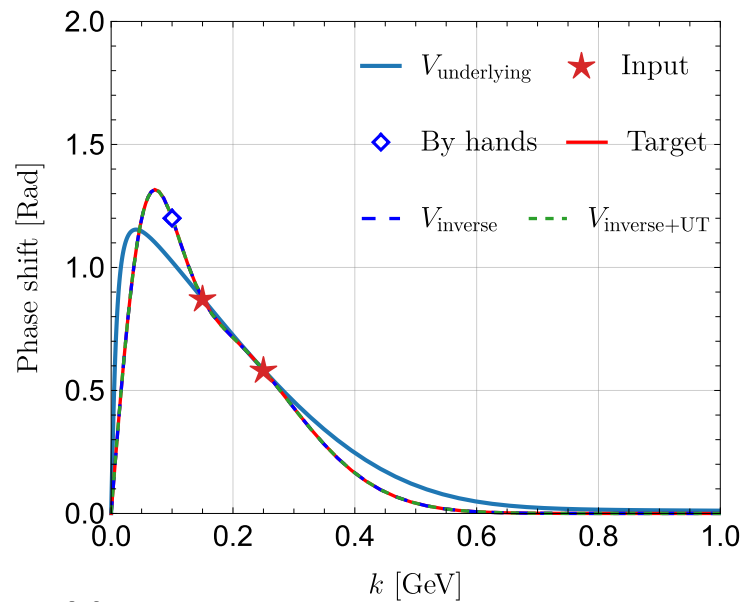
$$V^{inv+UT} = UV^{inv}U^\dagger + UH_0U^\dagger - H_0$$

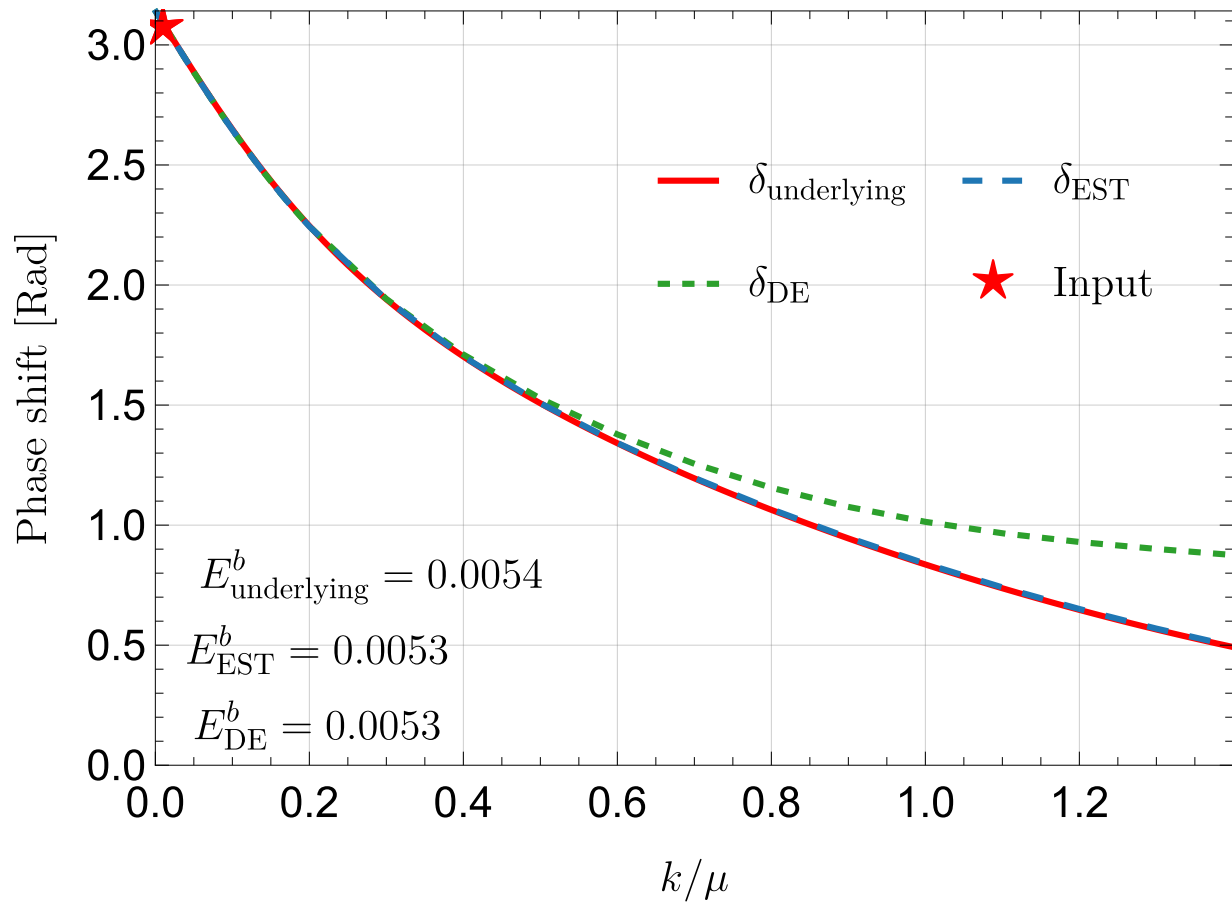


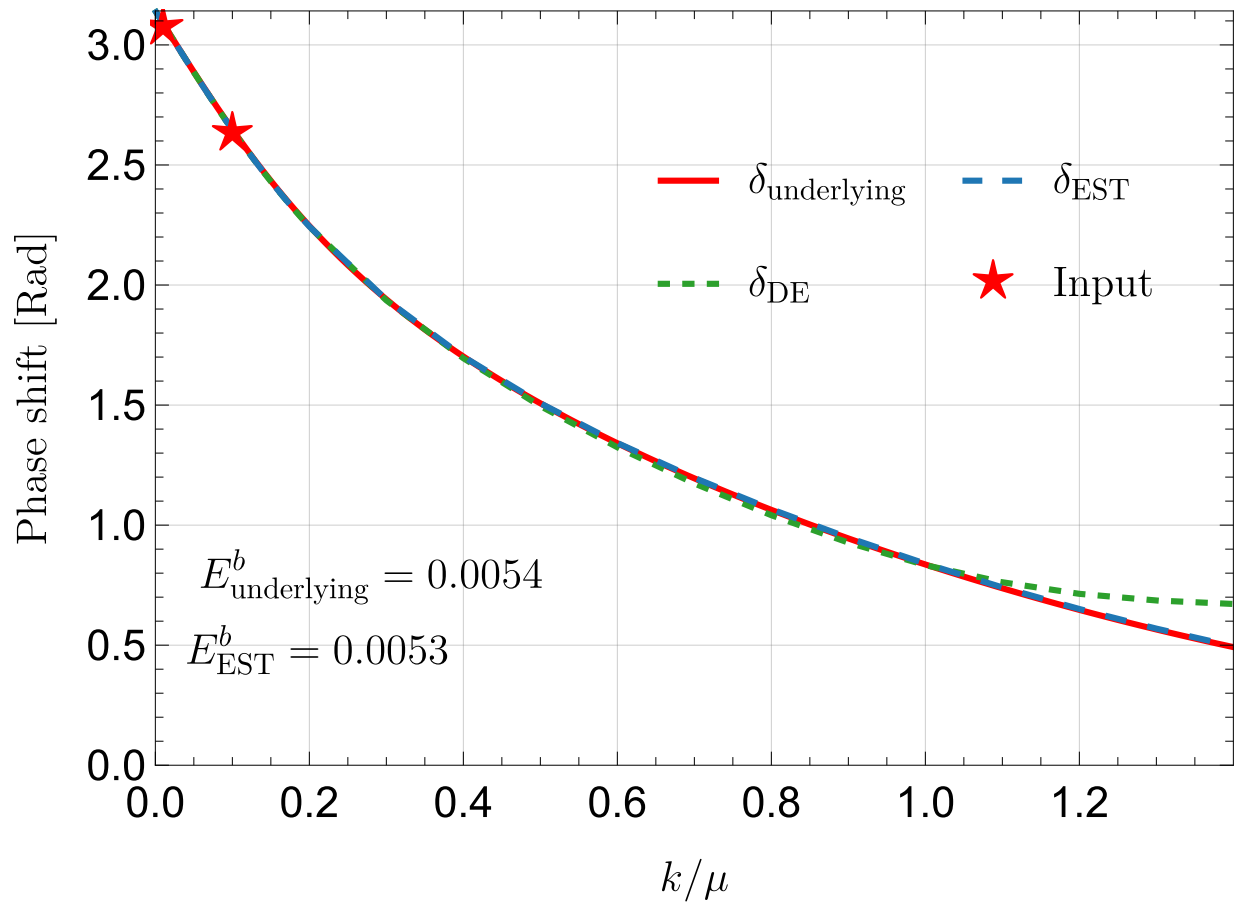
A small number of wave functions

- A small number of wave functions cannot fix the potential and phase shift
- Unless, you presume some features of potentials
 - ▶ Derivative expansion: the nonlocality of potential is small
 - ▶ EST: separable









A small number of wave functions

- Using two wave functions of $V_{underlying}$ as input $\{\psi_{k_1}(r), \psi_{k_2}(r)\}$
- $\delta_{tar}(k)$ go through $\{\delta(k_1), \delta(k_2)\}$ and the third phase shift $\delta_{by-hand}(k_3)$ assigned by hand
- Find a potential $V_{inverse}$ permit $\delta_{tar}(k)$ Tabakin:1969mr
- Find a unitary transformation give the correct wave functions $\{\psi_{k_1}(r), \psi_{k_2}(r)\}$ Ernst:1973utx
- Conclusion:
 - ▶ A small number of wave functions cannot fix the potentials and phase shifts
 - ▶ Unless, one presumes some features of potentials

