



山东大学  
SHANDONG UNIVERSITY

学无止境，气有浩然





山东大学（济南） | 26.06.2024

# Long-range interactions from the lattice QCD in a small box

Lu Meng (孟璐) | RUHR-UNIVERSITÄT BOCHUM

Base on [JHEP10\(2021\)051](#), and [arXiv:2312.01930](#)  
Together with V. Baru, E. Epelbaum, A. Filin, A.M. Gasparyan

- Lüscher's formula
  - ▶ Exponentially suppressed effect: left-hand cut problem
  - ▶ Partial wave mixing effect
  
- Our formalism
  
- $T_{cc}(3875)^+$  state
  
- Summary and outlook

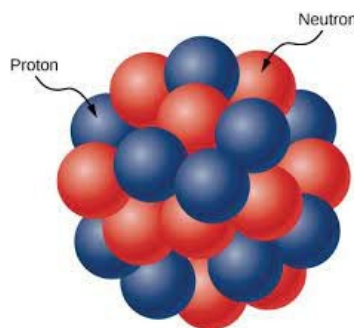
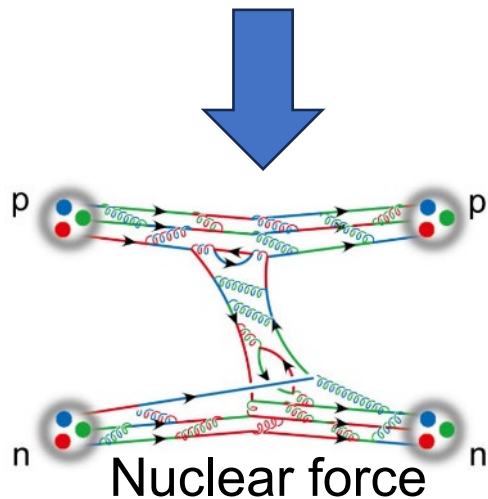
# From QCD to nuclei and neutron stars

$$\text{QCD: } \mathcal{L}_{QCD} = \sum_f \bar{q}_f (i\not{D} - \mathcal{M}_{q_f}) - \frac{1}{4} G_{\mu\nu}^a G^{\mu\nu,a}$$

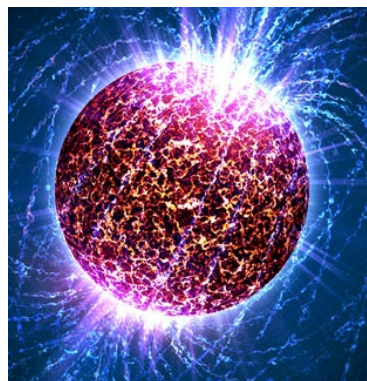
Phenomenological  
model



Chiral effective  
field theory



nucleus



neutron star

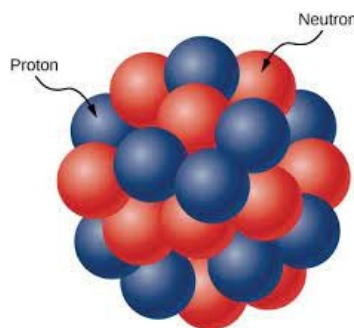
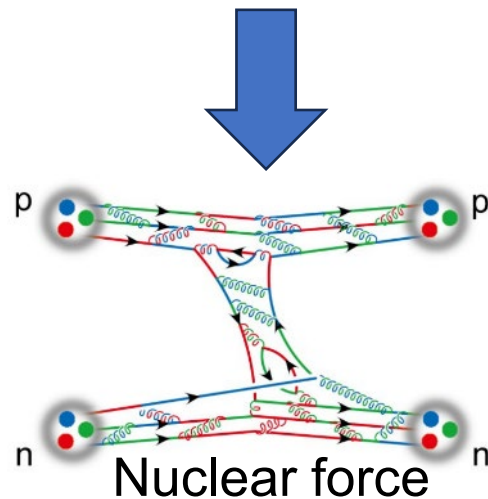
# From QCD to nuclei and neutron stars

$$\text{QCD: } \mathcal{L}_{QCD} = \sum_f \bar{q}_f (i\not{D} - M_{q_f}) - \frac{1}{4} G_{\mu\nu}^a G^{\mu\nu,a}$$

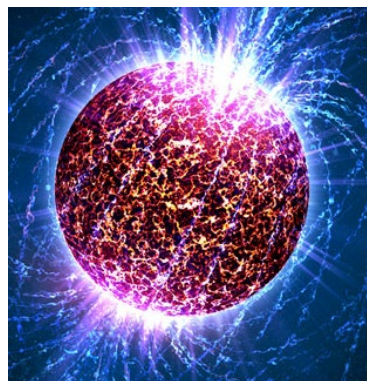
Phenomenological model



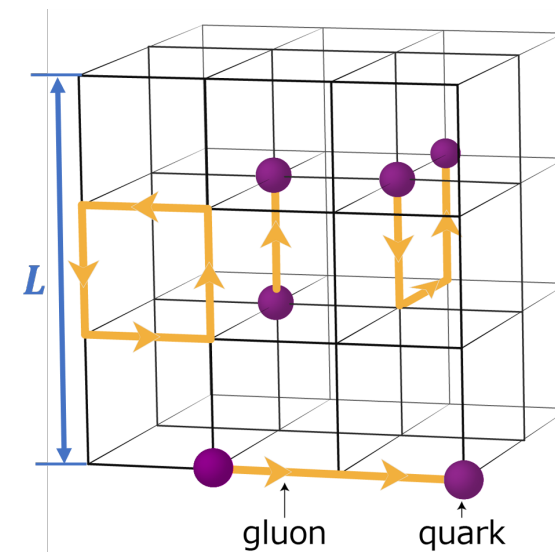
Chiral effective field theory



nucleus



neutron star

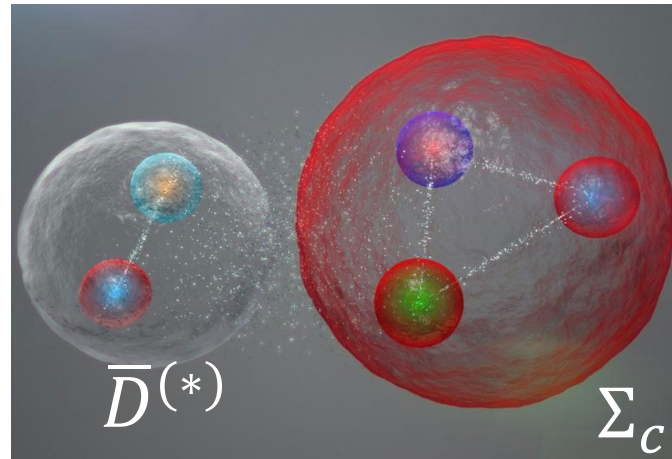


Lattice QCD

To extract observables in the infinite volume (IFV) from a FV calculation??

# Hadronic molecule

$$\text{QCD: } \mathcal{L}_{QCD} = \sum_f \bar{q}_f (i\not{D} - \mathcal{M}q_f) - \frac{1}{4} G_{\mu\nu}^a G^{\mu\nu,a}$$



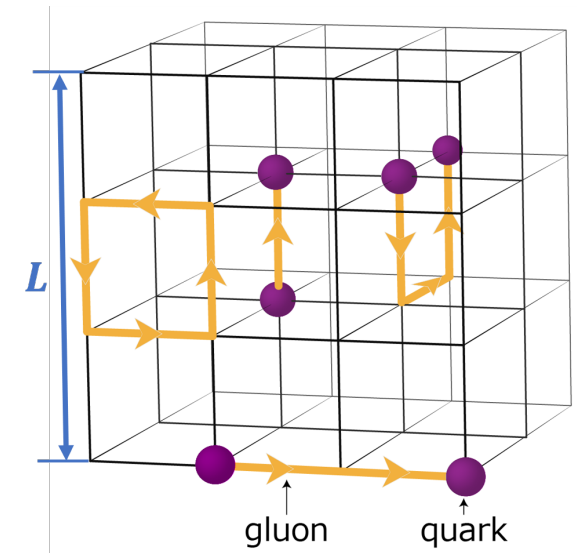
hadronic molecule

LHCb:2015yax

Phenomenological  
model



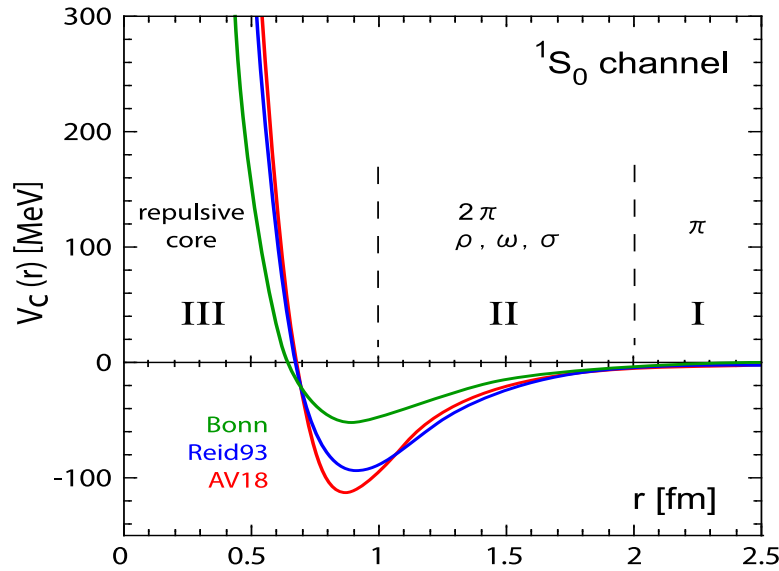
Chiral effective  
field theory



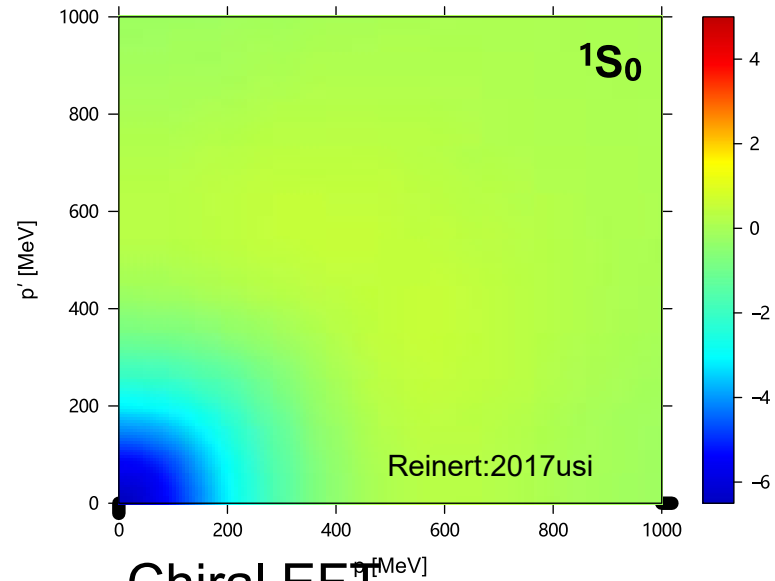
**Lattice QCD**

To extract observables in  
the infinite volume (IFV)  
from a FV calculation??

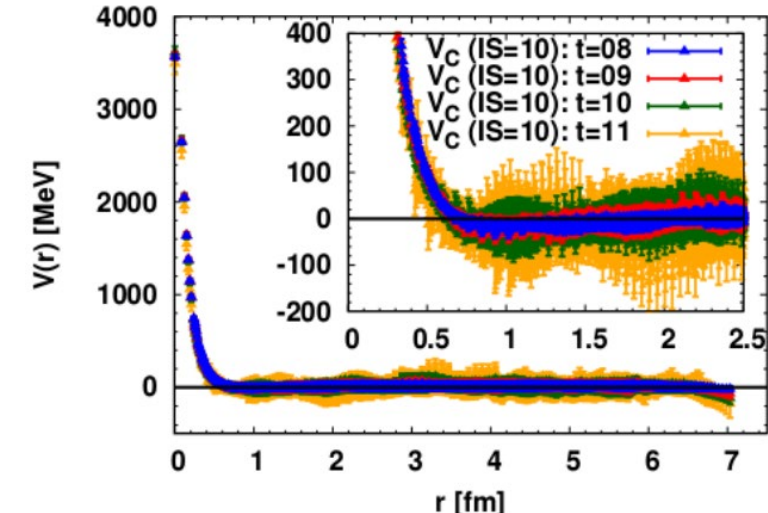
# Potential methods



Phenomenological model



Chiral EFT



HALQCD

- Raw data: Nambu-Bethe-Salpeter wave function (NBS WFs) [Ishii:2006ec](#), [Aoki:2009ji](#), [Aoki:2012tk](#)
- Get observables AKA: HALQCD (Hadrons to atomic nuclei from LQCD) method

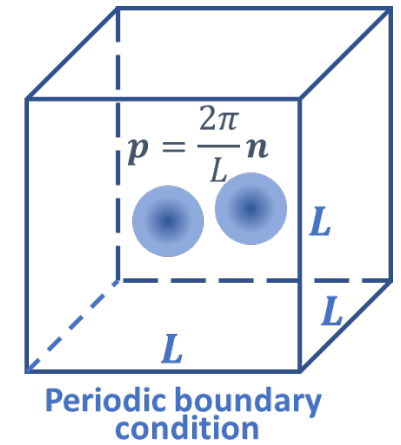
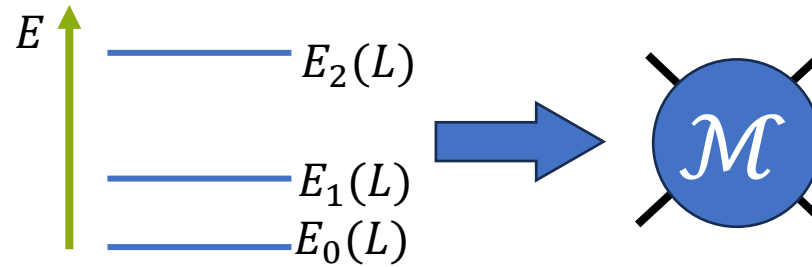


- ▶ Often criticized for **uncontrolled systematics**
- ▶ Potentials cannot be determined by the observables (phase shifts etc)

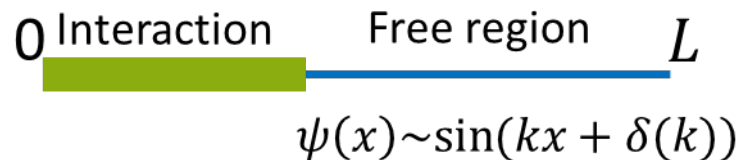
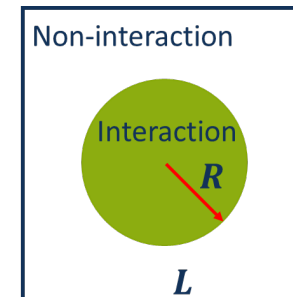
# Energy level method

- Raw data: finite volume energy levels  $E^{FV}$
- Get observables: Lüscher's formula
  - ▶  $E^{FV} \sim \delta(E^{FV})$

Lüscher:1990ux  
AKA: Lüscher method



- Requirement: larger box and short interaction  $\frac{L}{2} \gg R$
- Derivation example of Lüscher's formula
  - ▶ The  $(\nabla^2 + k^2)\psi(r) = 0$  for  $r > R$
  - ▶ Asymptotic behavior:  $\psi_k(r) \sim \frac{e^{i\delta(k)} \sin[kr + \delta(k)]}{kr}$
  - ▶ 1-D example

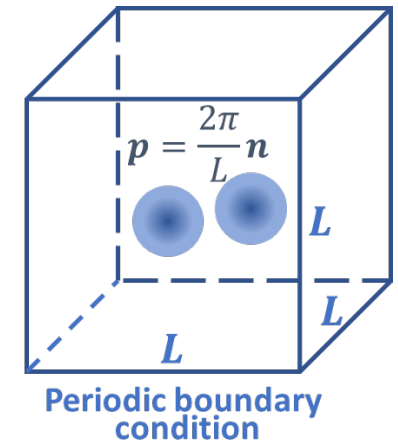
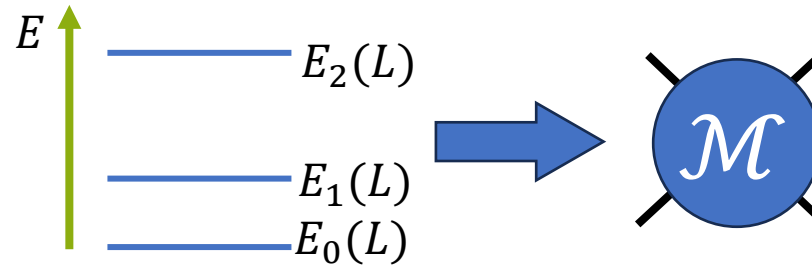


- ▶ Quantization condition:  $\sin(kL + \delta(k)) = 0 \Rightarrow kL + \delta(k) = n\pi$

# Energy level method

- Raw data: finite volume energy levels  $E^{FV}$
- Get observables: Lüscher's formula
  - ▶  $E^{FV} \sim \delta(E^{FV})$

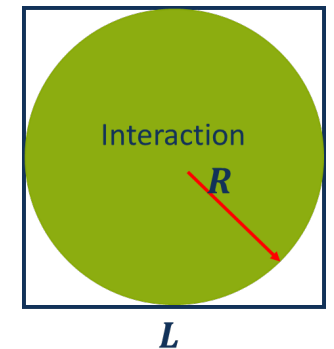
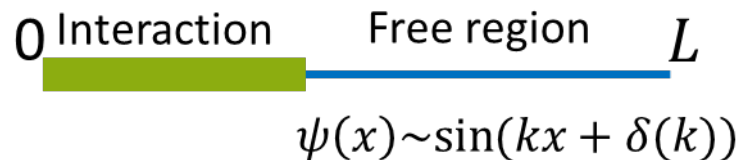
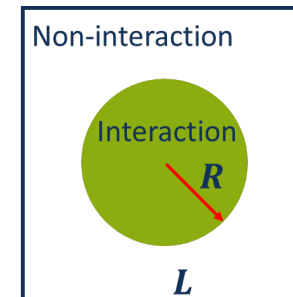
Lüscher:1990ux  
AKA: Lüscher method



- Requirement: larger box and short interaction  $\frac{L}{2} \gg R$

- Derivation example of Lüscher's formula

- ▶ The  $(\nabla^2 + k^2)\psi(r) = 0$  for  $r > R$
- ▶ Asymptotic behavior:  $\psi_k(r) \sim \frac{e^{i\delta(k)} \sin[kr + \delta(k)]}{kr}$
- ▶ 1-D example



Long-range interaction and small box???

- ▶ Quantization condition:  $\sin(kL + \delta(k)) = 0 \Rightarrow kL + \delta(k) = n\pi$

# Lüscher formula

- K-matrix:  $S_l = 1 + 2iT_l$ ,  $K_l^{-1} = T_l^{-1} + i$ 
  - ▶ Unitarity:  $S^\dagger S = 1$ , or  $S = e^{2i\delta_l}$ , or  $T^\dagger T = TT^\dagger = \text{Im } T$
  - ▶  $T^{-1} = \cot \delta_l - i$
  - ▶  $K_l^{-1} = \cot \delta_l$  is the real part of  $T_l^{-1}$

- Lüscher formula

$$G_F(L, k) - K_l^{-1}(k) = 0$$

AKA: Lüscher quantization conditions (LQCs)

For S-wave

kinematic term;  
Real object

K matrix in the IFV

$$G_F(L, k) = \frac{1}{\pi^2 n_k} Z_{00}(1, n_k), \quad Z_{00}(s, n_k) = \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{n} \in \mathbb{Z}^3} \frac{1}{(\mathbf{n}^2 - n_k^2)^s}$$

Input:  $E_{FV}$ , output  $\delta_l(E_{FV})$

$$\frac{2\pi}{L} n_k = k, \quad E_{FV} = 2\sqrt{m^2 + k^2}$$

- Two drawbacks
  - ▶ Exponentially suppressed effect
  - Left-hand cut problem
  - ▶ Partial wave mixing effect: cubic box

# Exponentially suppressed effect

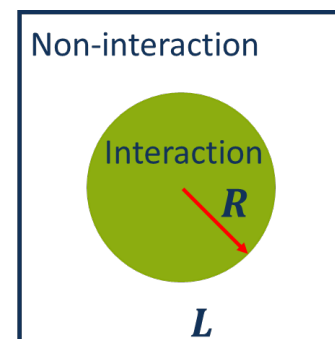
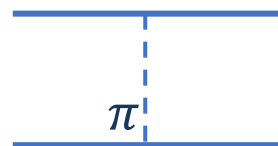
- Poisson summation formula

$$\frac{1}{L^3} \sum_{\vec{q}=2\pi\vec{n}/L} f(\vec{q}) = \int \frac{d^3q}{(2\pi)^3} f(\vec{q}) + \boxed{\sum_{\vec{n} \neq 0, \vec{n} \in \mathbb{Z}^3} \int \frac{d^3q}{(2\pi)^3} f(\vec{q}) e^{iL\vec{q} \cdot \vec{n}}}$$

▶ E.g  $f(\vec{q}) = \frac{1}{m+q^2}$ , the second term:  $\frac{e^{-mLn}}{mLn}$

Exponentially suppressed

- Long-range interaction: one-pion exchange interaction



- Discrepancy of the Lüscher's formula

▶ Exponentially suppressed effect:  $e^{-m_\pi L}$

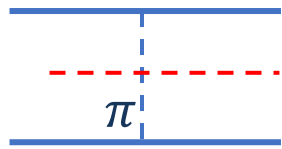
▶ Requirement:  $\frac{L}{2} \gg R$ , typically  $m_\pi L > 3 \Rightarrow L > 4.3$  fm

▶ A practical box size: 2–4 fm

- A special exponentially suppressed effect: left-hand cut problem

# Left-hand cut problems

- Left-hand cut (lhc) from the one-pion exchange interaction



Left-hand cut

$$V(\vec{p}, \vec{p}') = \frac{1}{(\vec{p}' - \vec{p})^2 + m^2}$$

- Partial wave decomposition, e.g. S-wave

$$V_{l=0}(p, p') = \int_{-1}^1 dz \frac{1}{p^2 + p'^2 - 2pp'z + m^2} = -\frac{1}{2pp'} \log \left( \frac{(p - p')^2 + m^2}{(p + p')^2 + m^2} \right)$$

- On-shell  $p = p' = k$ ,  $k^2 = 2\mu E$

$$V_{l=0}(k, k) = \int_{-1}^1 dz \frac{1}{2k^2(1 - z) + m^2}$$

$$2k^2(1 - z) + m^2 = 0 \Rightarrow z = \frac{m^2}{2k^2} + 1$$

on-shell pion

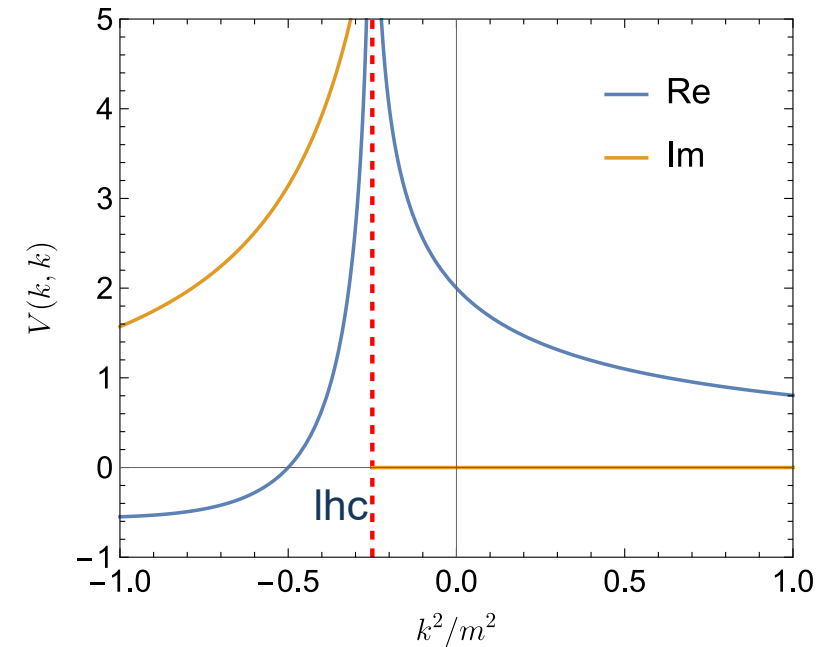
$$-1 < z < 1 \Rightarrow k^2 < -\frac{m^2}{4}$$

- ▶ Branch point:  $k^2 < -\frac{m^2}{4}$

- K-matrix

$$K = V + VG^{\mathcal{P}}K$$

- ▶ The imaginary part arises



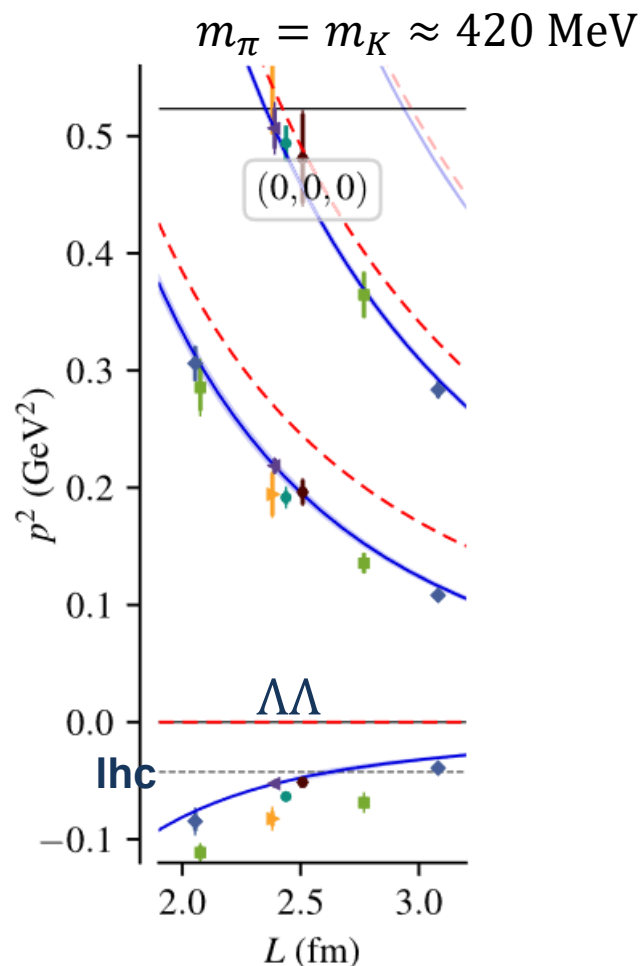
$$\det [G_F^{-1}(L, E) - K(E)] \neq 0$$

real       $\text{Im } K \neq 0$  for  $p^2 < -\frac{m^2}{4}$

Qualitative discrepancy

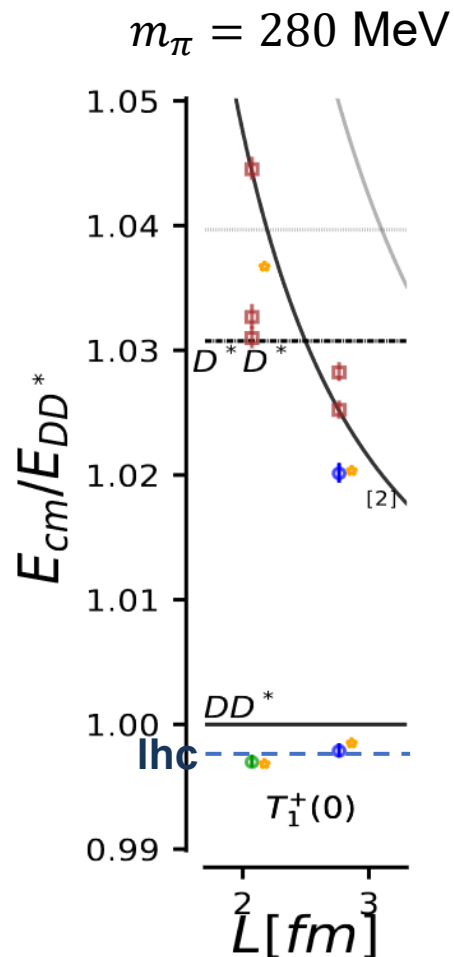
# Left-hand cut problems

Green:2021qol



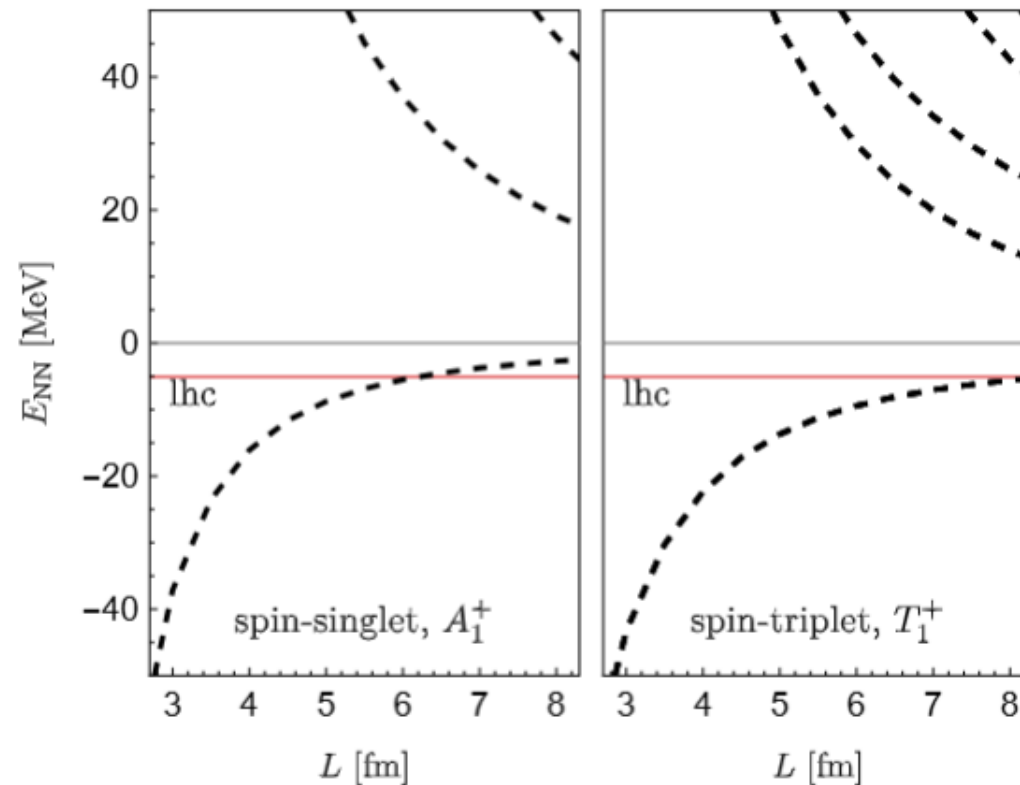
H-dibaryons ( $udsuds$ )

Padmanath:2022cvl



$DD^*$  ( $T_{cc}$ )

$m_\pi = 137$  MeV



NN systems

$L > 8$  fm to get rid of the lhc problem

# Left-hand cut problem

Lattice 2023

<https://indico.fnal.gov/event/57249/>

Finite-volume scattering on the left-hand cut

Andre Baião Raposo



Curia II, WH2SW

13:30 - 13:50

Breakdown of Lüscher Formalism near Left Hand Cuts

Md Habib E Islam



Curia II, WH2SW

13:50 - 14:10

Resolving the left-hand-cut problem in lattice studies of the doubly-charmed tetraquark

Steve Sharpe



Curia II, WH2SW

14:10 - 14:30

Finite-volume scattering on the left-hand cut

André Baião Raposo, Maxwell T. Hansen

Nov 30, 2023

59 pages

e-Print: [2311.18793](#) [hep-lat]

Modified Lüscher QCs  
framework and toy models

Solving the left-hand cut problem in lattice QCD:  $T_{cc}(3875)^+$  from finite volume energy levels

Lu Meng, Vadim Baru, Evgeny Epelbaum, Arseniy A. Filin, Ashot M. Gasparyan

Dec 4, 2023

13 pages

e-Print: [2312.01930](#) [hep-lat]

EFT in FV

Application in  $T_{cc}$

Lüscher equation with long-range forces

Rishabh Bubna (Bonn U. and Bonn U., HISKP), Hans-Werner Hammer (Darmstadt, Tech. U. and Darmstadt, EMMI and FAIR, Darmstadt), Fabian Müller (Bonn U. and Bonn U., HISKP), Jin-Yi Pang (Shanghai U. Sci. Tech.), Akaki Rusetsky (Bonn U. and Bonn U., HISKP and Tbilisi State U.) [Show All\(6\)](#)

Feb 20, 2024

28 pages

Published in: *JHEP* 05 (2024) 168

Published: May 14, 2024

e-Print: [2402.12985](#) [hep-lat]

# Quantization of momentum

- Boundary conditions in the cubic box

$$\psi(\mathbf{x}_1, \mathbf{x}_2) = \psi(\mathbf{x}_1 + \mathbf{n}_1 L, \mathbf{x}_2 + \mathbf{n}_2 L)$$

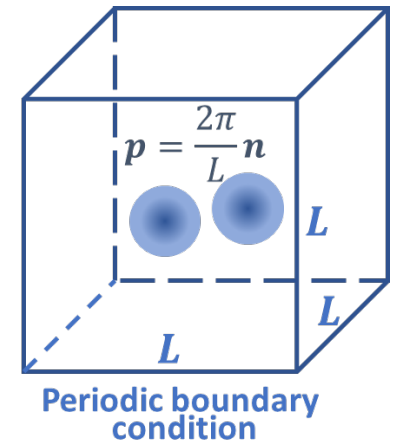
$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{P}, \quad \mathbf{p}_1 = \frac{2\pi}{L} \mathbf{n}, \quad \mathbf{P} = \frac{2\pi}{L} \mathbf{d}, \quad \mathbf{n}, \mathbf{d} \in Z^3$$

- ▶ 2-body rest systems:  $\mathbf{d} = (0,0,0)$
- The rotation symmetry is broken:  $SO(3) \rightarrow O_h$ 
  - ▶  $\{l, m\}$  are not good quantum numbers to label states
  - ▶ Partial wave mixing, for  $l \neq l'$  and  $m \neq m'$ ,
- ▶ The FV energy should be classified by irreducible representations (irreps.) of  $O_h$

$$\langle lm | H^{FV} | l' m' \rangle \neq 0$$

- Moving system in the box  $\mathbf{P} = \frac{2\pi}{L} \mathbf{d} \neq 0$

- ▶ Other point groups,  $D_{4h}, D_{2h} \dots$
- ▶ If  $m_1 \neq m_2$ , space inversion invariance is broken; states with different parities could mix



- Expanding it in partial wave (PW) basis

$$\det[G_F - K^{-1}] = 0, \Rightarrow \det[M_{l'm',lm} - \delta_{ll'}\delta_{mm'} \cot \delta_l] = 0$$

- ▶ Determinate equation of a matrix with infinite dimensions.
- ▶ Truncate at some  $l_{\max}$
- Reduce to irreps.  $\Gamma_i$  of point group::  $\det \left[ M_{ln,l'n'}^{(\Gamma,P)} - \delta_{ll'}\delta_{nn'} \cot \delta_l \right] = 0$
- Example  $\Gamma = A_1^+$ ,  $w_{lm}$  depends on  $E$  but independent on  $V$

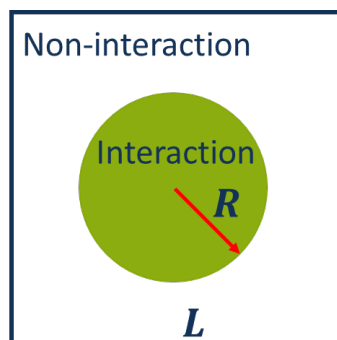
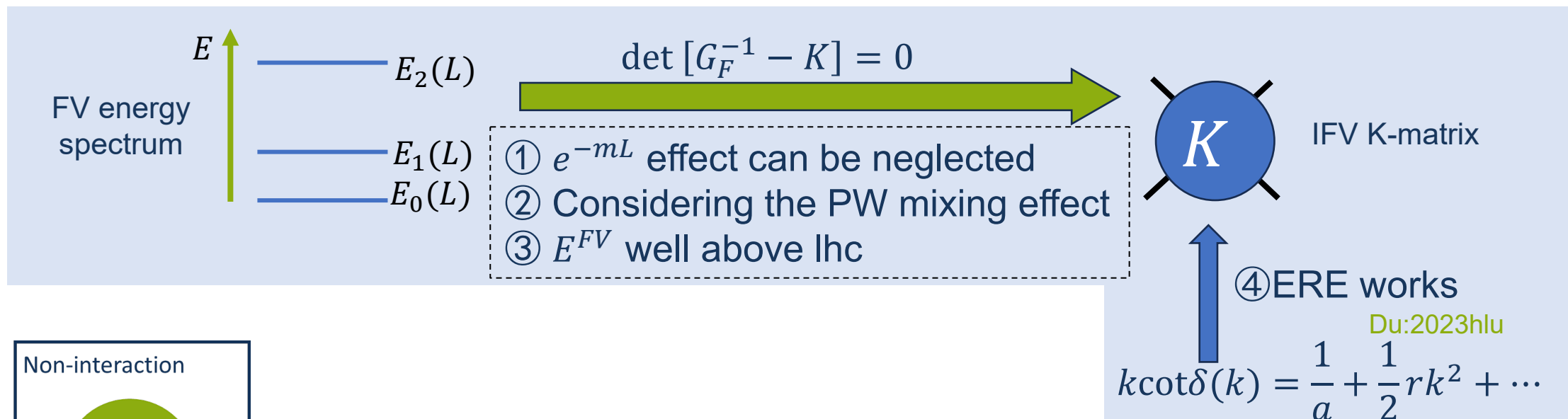
$$\det \left[ M_{ln,l'n'}^{(\Gamma,P)} - \delta_{ll'}\delta_{nn'} \cot \delta_l \right] = 0, \quad M^{(A_1^+,d)} = \begin{bmatrix} w_{00} & -\sqrt{5}w_{20} & \cdots \\ -\sqrt{5}w_{20} & w_{00} + \frac{10}{7}w_{20} + \frac{18}{7}w_{40} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Bernard:2008ax

- Truncate at  $l_{\max} = 0$ , one-to-one relation:  $\delta_0(E^{FV}) \sim E^{FV}$
- Truncate at  $l_{\max} > 0$ , no one-to-one relation
  - ▶ E.g.  $\{E_1^{FV}, E_2^{FV}\} \not\Rightarrow \{\delta_S(E_1^{FV}), \delta_S(E_2^{FV}), \delta_D(E_1^{FV}), \delta_D(E_2^{FV}) \dots\}$
  - ▶ One has to parameterize the K-matrix: e.g. effective range expansions (ERE)

Luscher:1990ux,Rummukainen:1995vs,Feng:2004ua, Kim:2005gf,Fu:2011xz,Polejaeva:2012ut,Leskovec:2012gb,Gockeler:2012yj,...

# Requirements of a practical Lüscher method



Requirement:  $\frac{L}{2} \gg R$   
 Typically:  $m_\pi L > 3$

High partial wave suppression

- Threshold effect:  $T_l(p) \sim p^{2l}$
- The large scale:  $m_\pi$

Taylor's textbook P197



All four requirements constrained by the  $V_{1\pi}$

---

# **Our strategy**

# Solving the lhc problem

- Our strategy: solve the Schrödinger Eq. to get the **bound state** solutions

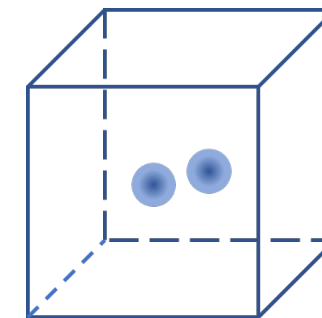
$$\frac{\mathbf{p}^2}{2\mu} \psi(\mathbf{p}) + \int \frac{d^3\mathbf{p}'}{(2\pi)^3} V(\mathbf{p}, \mathbf{p}') \psi(\mathbf{p}') = E \psi(\mathbf{p})$$

Off-shell, for  $E < 0$

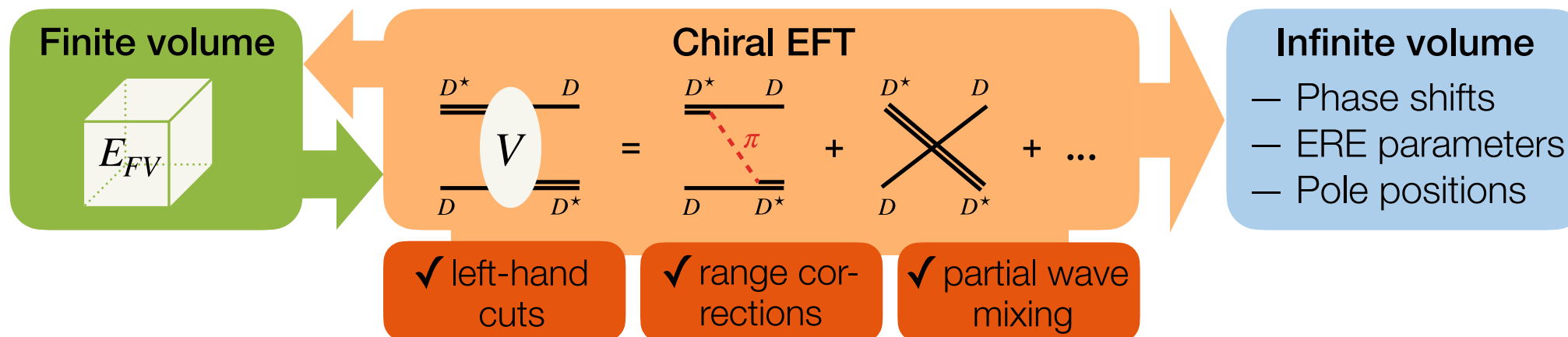
- ▶ Works well even for  $2\mu E < -\frac{m^2}{4}$
- ▶ In the finite volume:

$$\int \frac{d^3\mathbf{p}'}{(2\pi)^3} \rightarrow \frac{1}{L^3} \sum_{\mathbf{p}_n}$$

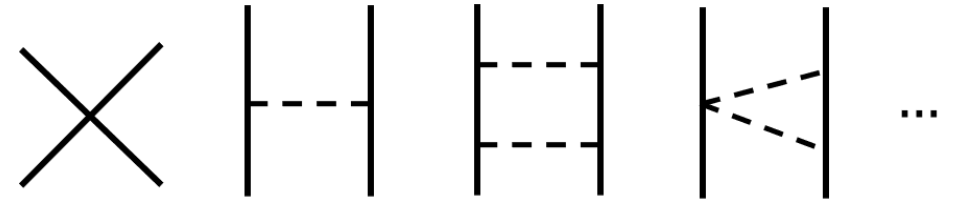
- ▶ Include the OPE explicitly
- ▶ Plane wave basis with discrete momentum: cubic group symmetry



$E_{FV}$  are “bound states”  
trapped by the potential well



$$V(\vec{p}', \vec{p}) = V_{\text{contact}} + V_{1\pi} + V_{2\pi}$$

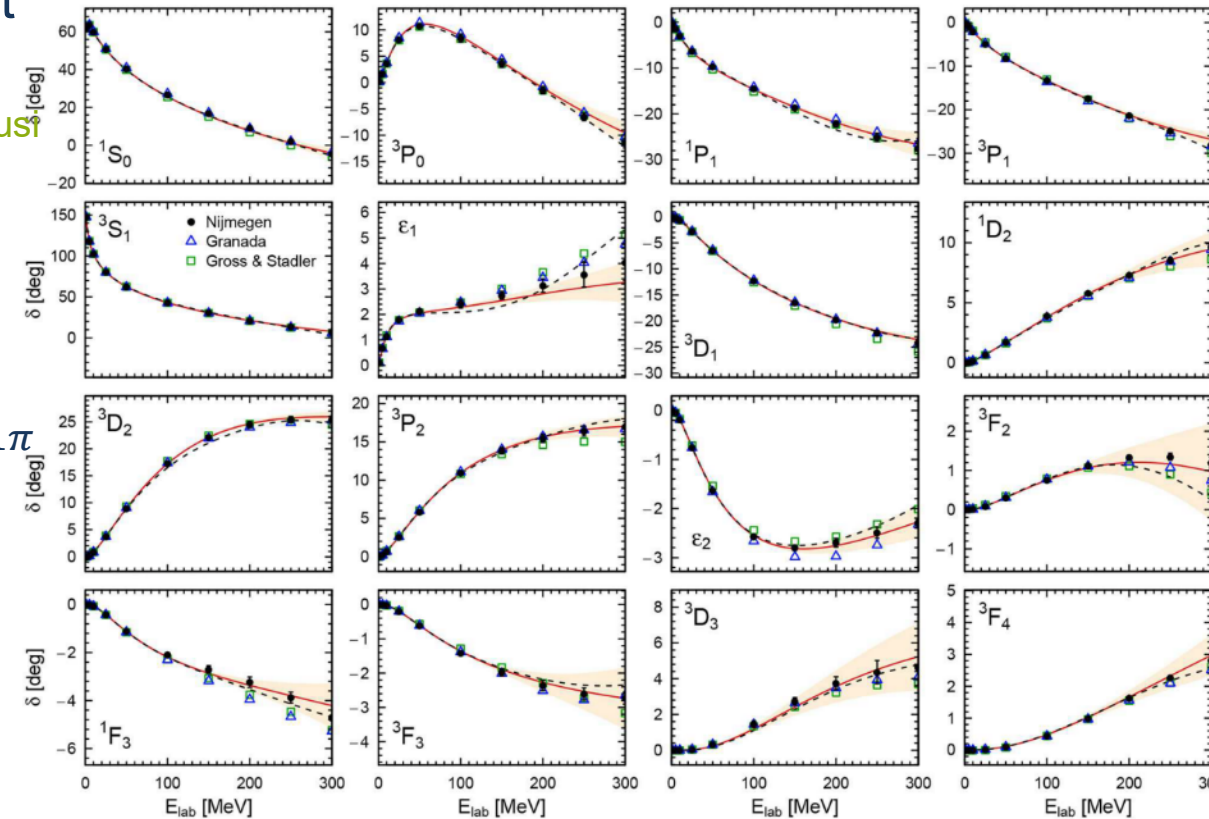


- Derived in the momentum space,  $E$ -independent
- Semilocal momentum-space regularization

Reinert:2017ust

$$V_{1\pi}(\vec{p}', \vec{p}) = -\frac{g_A^2}{4F_\pi^2} \left( \frac{\vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q}}{q^2 + m_\pi^2} + C(m_\pi) \vec{\sigma}_1 \cdot \vec{\sigma}_2 \right) e^{-\frac{q^2 + m_\pi^2}{\Lambda^2}}$$

- Benefit from the known long-range interaction  $V_{1\pi}$
- Low energy constants (LECs) for short-range interaction (contact interaction)
  - ▶ fitting lattice QCD data



# Our formalism: plane wave basis expansion

- $|\mathbf{p}_n, \boldsymbol{\eta}\rangle$ :  $\mathbf{p}_n$  discrete momentum,  $\boldsymbol{\eta}$ : polarization vector for  $S = 1$

$$\hat{D}(g)|\mathbf{p}, \boldsymbol{\eta}\rangle = |g\mathbf{p}, g\boldsymbol{\eta}\rangle, \hat{P}|\mathbf{p}, \boldsymbol{\eta}\rangle = |-\mathbf{p}, \boldsymbol{\eta}\rangle, \langle \mathbf{p}_{n'}, \boldsymbol{\eta}'^\dagger | \hat{D}(g) | \mathbf{p}_n, \boldsymbol{\eta} \rangle = \delta_{n'n} (\boldsymbol{\eta}'^\dagger \cdot g\boldsymbol{\eta})$$

- $\{|\mathbf{p}_n, \boldsymbol{\eta}\rangle\}$  form the representation space of corresponding point group
- For non-relativistic systems, Lippmann-Schwinger equation (LSE)

- ▶ matrix equation  $\mathbb{T} = \mathbb{V} + \mathbb{V}\mathbb{G}\mathbb{T}$
- ▶ Finite volume levels  $\Rightarrow$  Eigenvalue problem

$$\det(\mathbb{G}^{-1} - \mathbb{V}) = 0 \rightarrow \det(\mathbb{H} - E\mathbb{I}) = 0,$$

- ▶ Reduce the  $\mathbb{H}$  according to irreducible representations (irreps) of the point group

$$\mathbb{H} \Rightarrow \text{diag}\{\mathbb{H}_{\Gamma_i}, \mathbb{H}_{\Gamma_j}, \dots\} \Rightarrow \mathbb{H}_{\Gamma} \mathbf{v} = E_{\Gamma} \mathbf{v}$$

- For moving systems, elongated boxes, particles with arbitrary spin...
- dim of the  $\mathbb{H}_{\Gamma}$ : cubic function

$$\text{dim} \sim \left( \frac{\Lambda_{\text{UV}}}{2\pi/L} \right)^3 \sim \mathcal{O}(1000), \quad \Lambda_{\text{UV}} \approx 4\text{GeV}$$

Similar approaches: Momentum lattice ([Doring:2011ip](#)), Hamiltonian EFT([Wu:2014vma](#), [Liu:2015ktc](#))...

# Towards a practical approach: eigenvector continuation

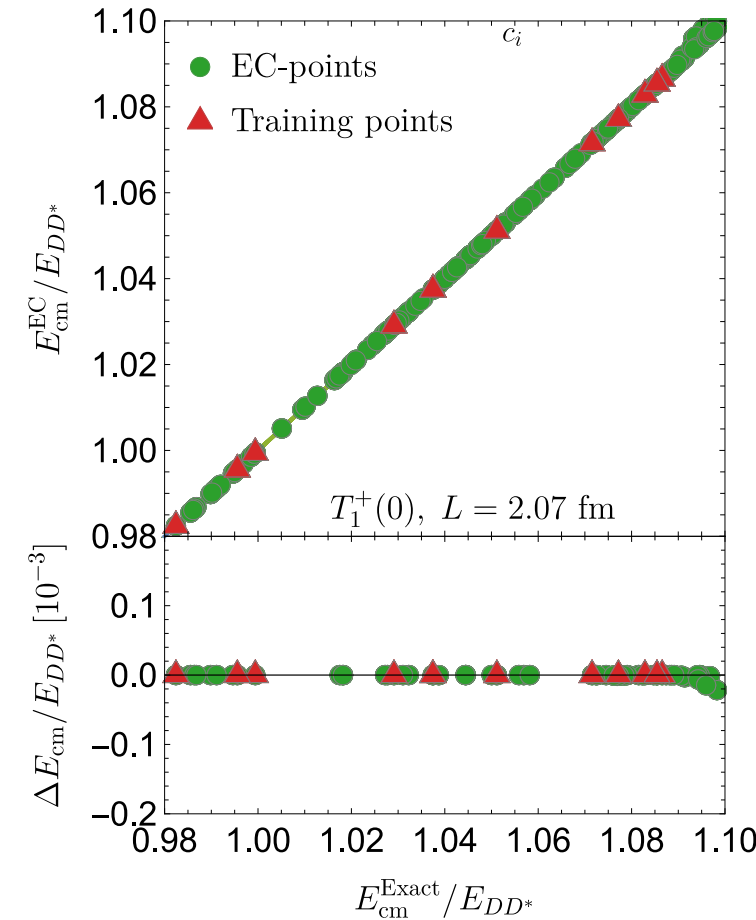
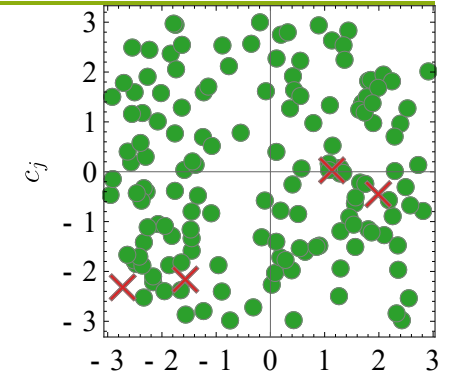
- Plane wave basis+Eigenvector continuation
  - ▶ Eigenvector continuation (EC) with subspace learning
- To fit or quantify uncertainty: solve eigenvalue problem with different  $\{c_i\}$  repeatedly
- EC basis: eigenvectors from a selection of parameter sets  $\{c_i\}_1, \{c_i\}_2, \dots$  (training point)
- Naturalness of LEC in EFT ( $\sim 1$ ) makes the EC more reliable
- dim is linear function

Frame:2017fah, Demol:2019yjt,  
Furnstahl:2020abp, Yapa:2022nnv

$$\dim^{EC} = \frac{p_{max}}{2\pi/L} \sim \mathcal{O}(10), \quad p_{max} \approx 0.6 \text{ GeV}$$

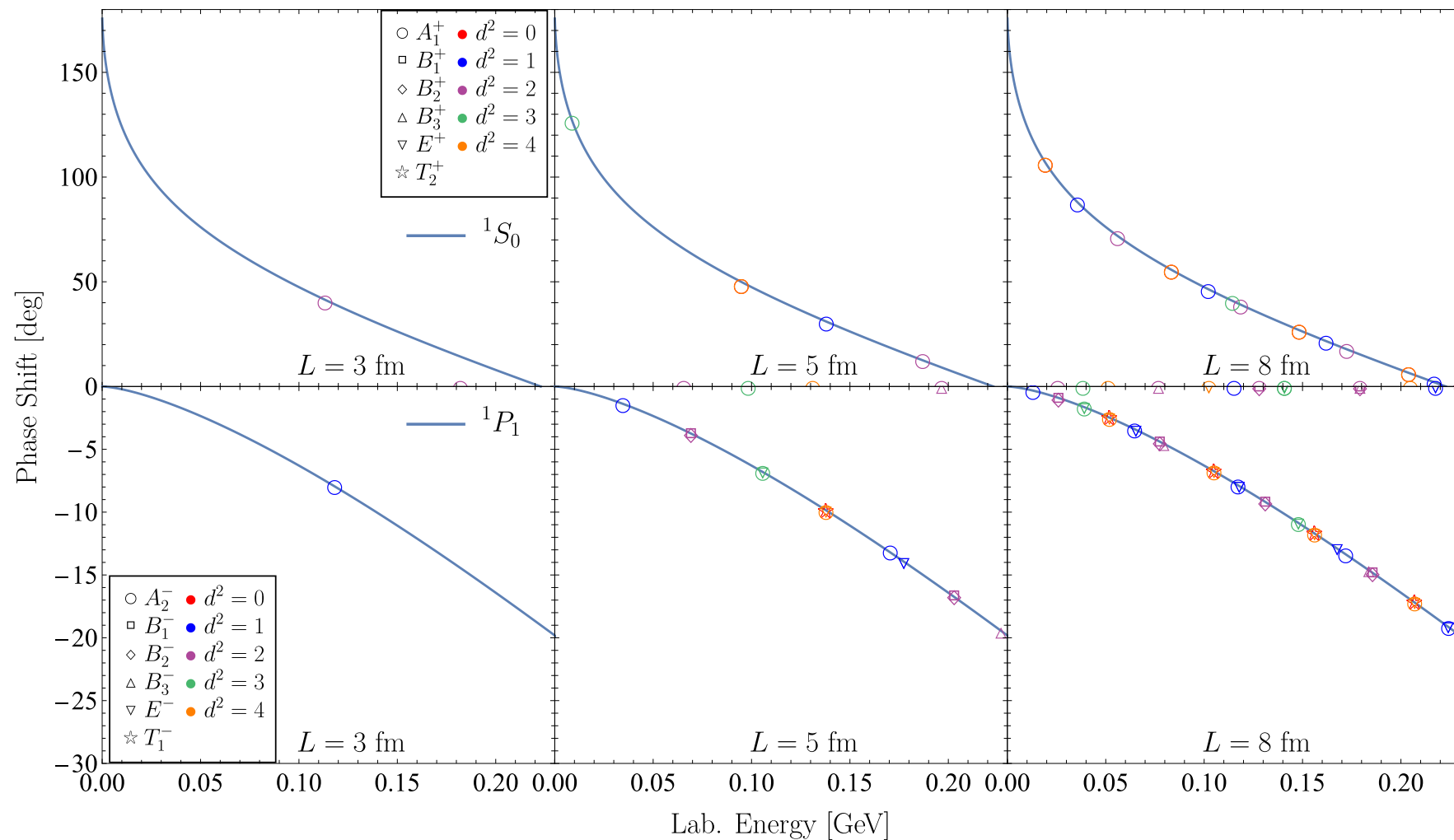
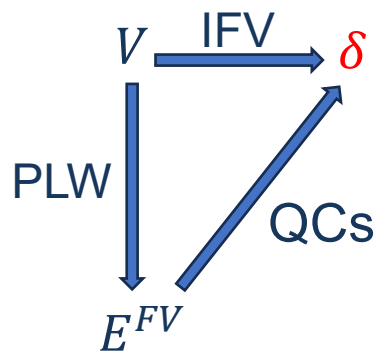
- The subspace learning is the one-time cost
- Make the calculation fast and accurate
- Easy-to-use interface : no need to know the detail of ChEFT

$$\mathbb{H}^{EC} = \mathbb{H}_0^{EC} + c_i \mathbb{V}_i^{EC}, \quad \mathbb{H}^{EC} \mathbf{v} = E \mathbf{v}$$



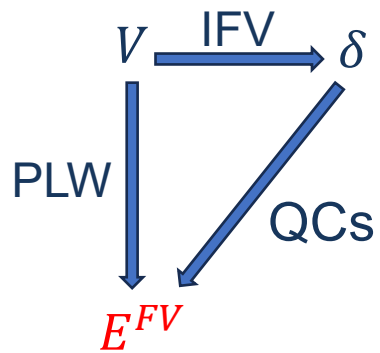
# Benchmark: contact interaction

- Contact interaction:  $V(\mathbf{p}, \mathbf{p}') = C_S + C_1 \mathbf{q}^2 + C_2 \mathbf{k}^2$
- Only contribute to S-wave and P-wave

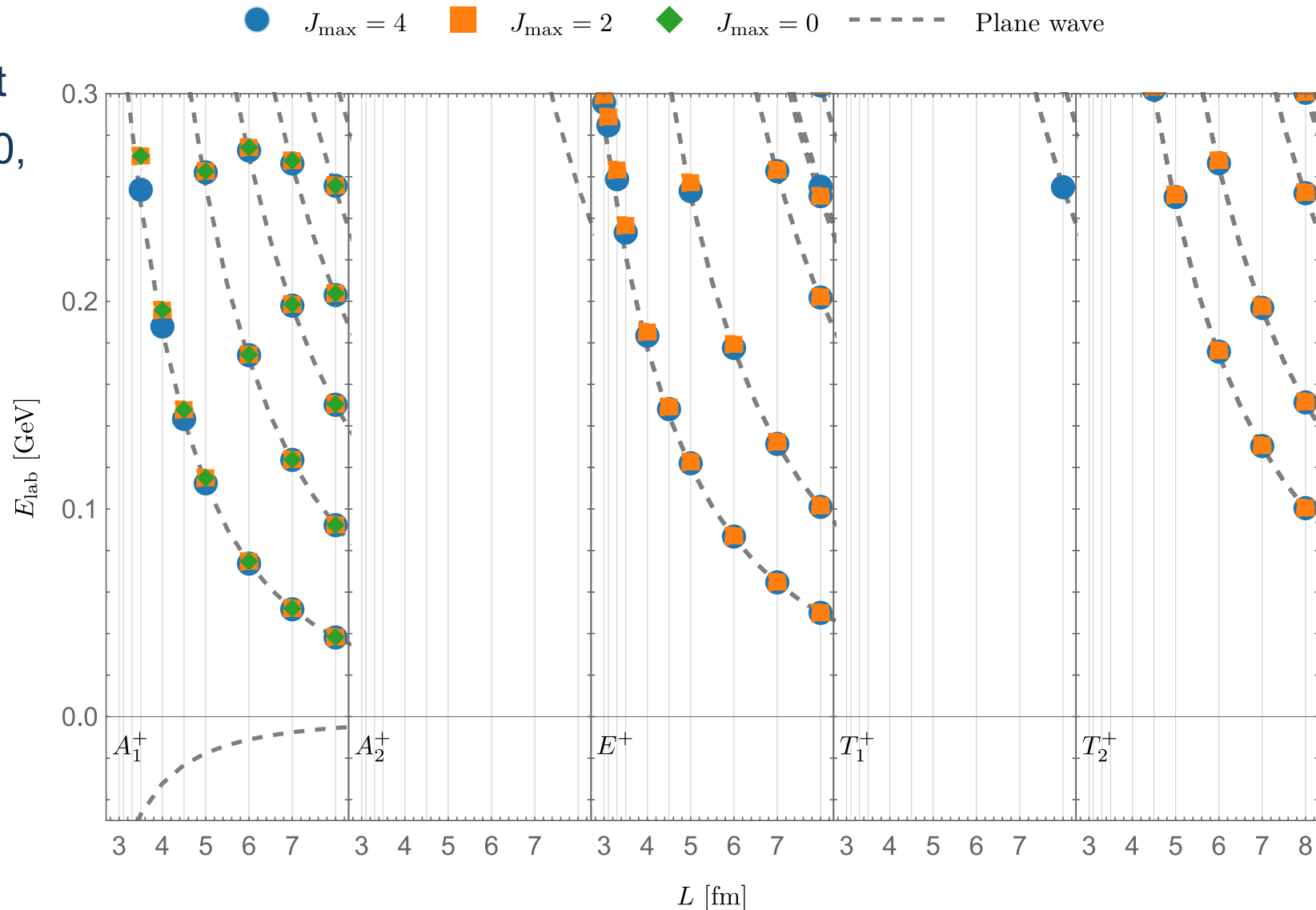


# Benchmark: chiral EFT

- ChEFT nuclear force: NNLO
- $S=0$ ,  $d = (0,0,0)$ , even parity
- QCs with partial mixing effect
- $L = \{ 3.0, 3.1, 3.3, 3.5, 4.0, 4.5, 5.0, 6.0, 7.0, 8.0 \}$  fm

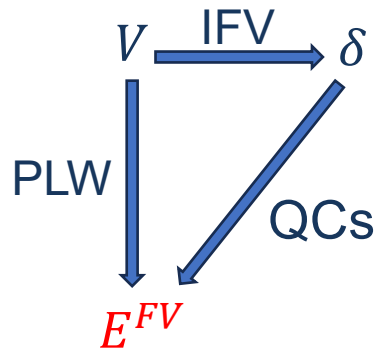


- The discrepancy
  - ▶ Small box
  - ▶ Small  $J_{\max}$  truncation

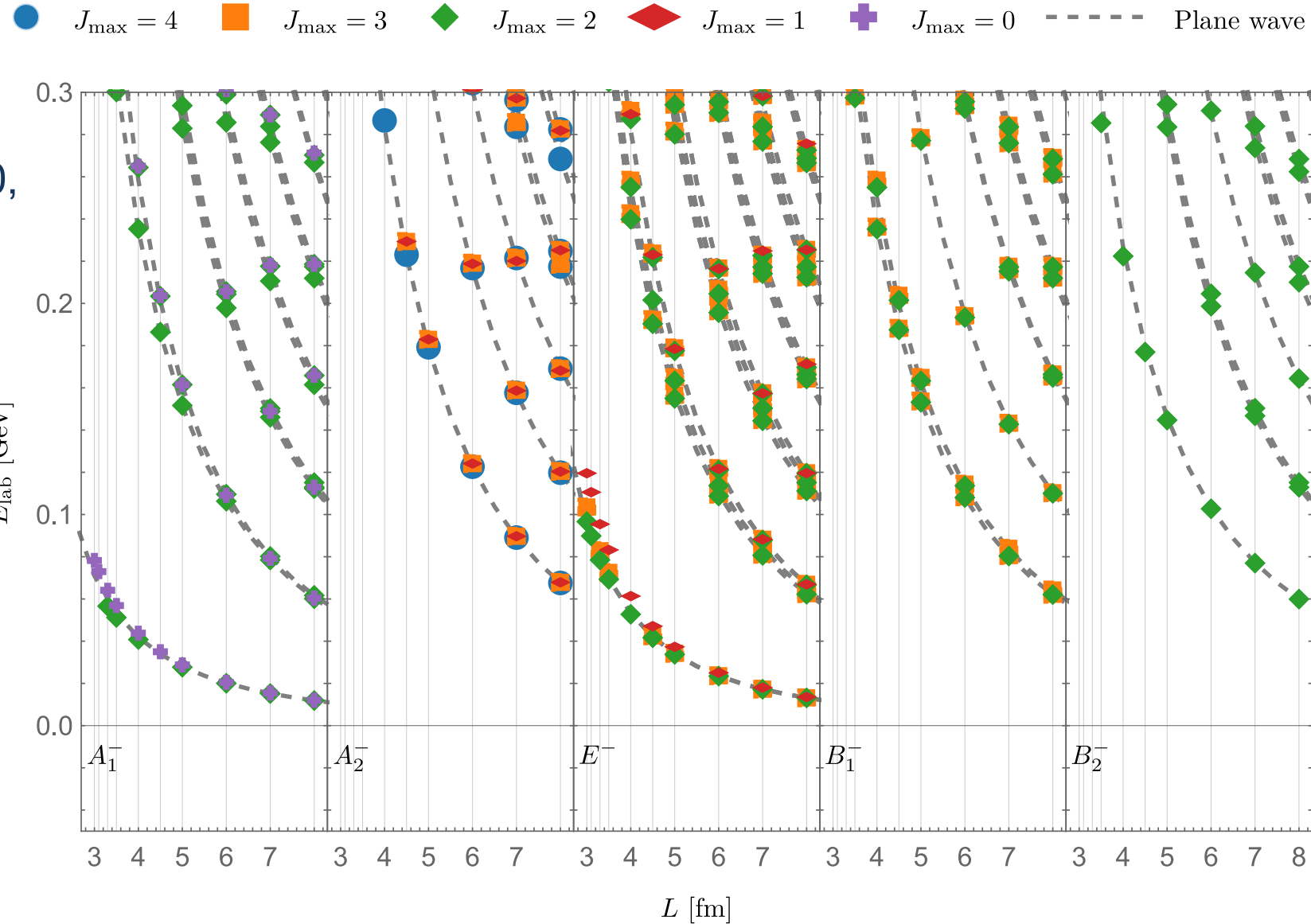


# Benchmark: chiral EFT

- ChEFT nuclear force: NNLO
- $S=1$ ,  $d=(0,0,1)$ , odd parity
- QCs with partial mixing effect
- $L=\{3.0,3.1,3.3,3.5,4.0,4.5,5.0,6.0,7.0,8.0\}$  fm



- The discrepancy
  - ▶ Small box
  - ▶ Small  $J_{\max}$  truncation



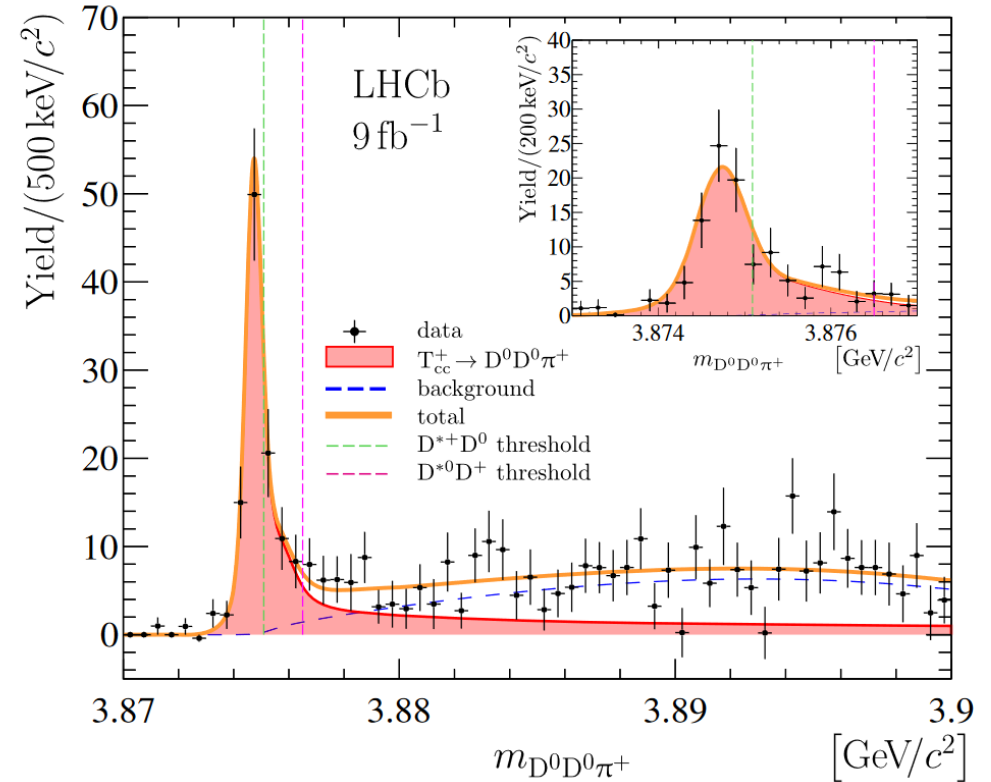
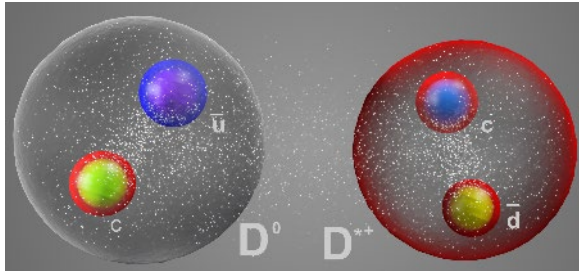
---

$T_{cc}(3875)^+$  **state**

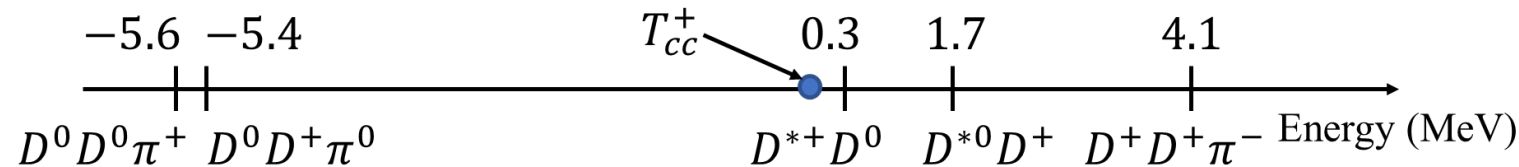
# $T_{cc}(3875)^+$ state

LHCb Collaboration

- $T_{cc}(3875)^+$  was observed in 3-body final states:  $D^0 D^0 \pi^+$
- Very close to  $D^0 D^{*+}$  thresholds:  $\delta m_U \approx -360 \text{ keV}$ ,  $\Gamma \approx 48 \text{ keV}$
- Exotic hadrons: minimal quark content:  $cc\bar{u}\bar{d}$
- Good candidates of  $D^0 D^{*+}$  molecule



- 3-body dynamics could be important



LHCb:2021vvq, LHCb:2021auc, Du:2021zzh, Meng:2021jnw...

# $T_{cc}$ lattice QCD simulations

Padmanath:2022cvi

● LQCD:  $m_\pi \approx 280$  MeV,  $m_D \approx 1927$  MeV,  $m_{D^*} \approx 2049$  MeV,  $L \approx 2.07, 2.76$  fm,  $a \approx 0.086$  fm

● Some quick estimations

▶  $m_{\text{eff}}^2 = m_\pi^2 - (m_{D^*} - m_D)^2 > 0, m_{\text{eff}} \approx 252$  MeV

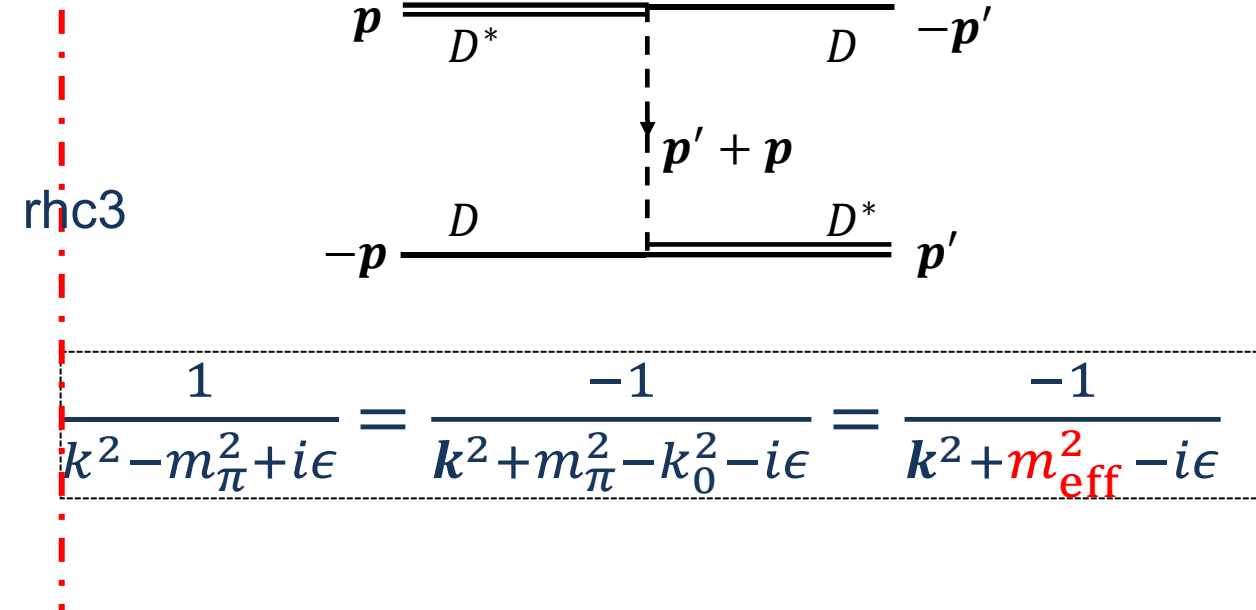
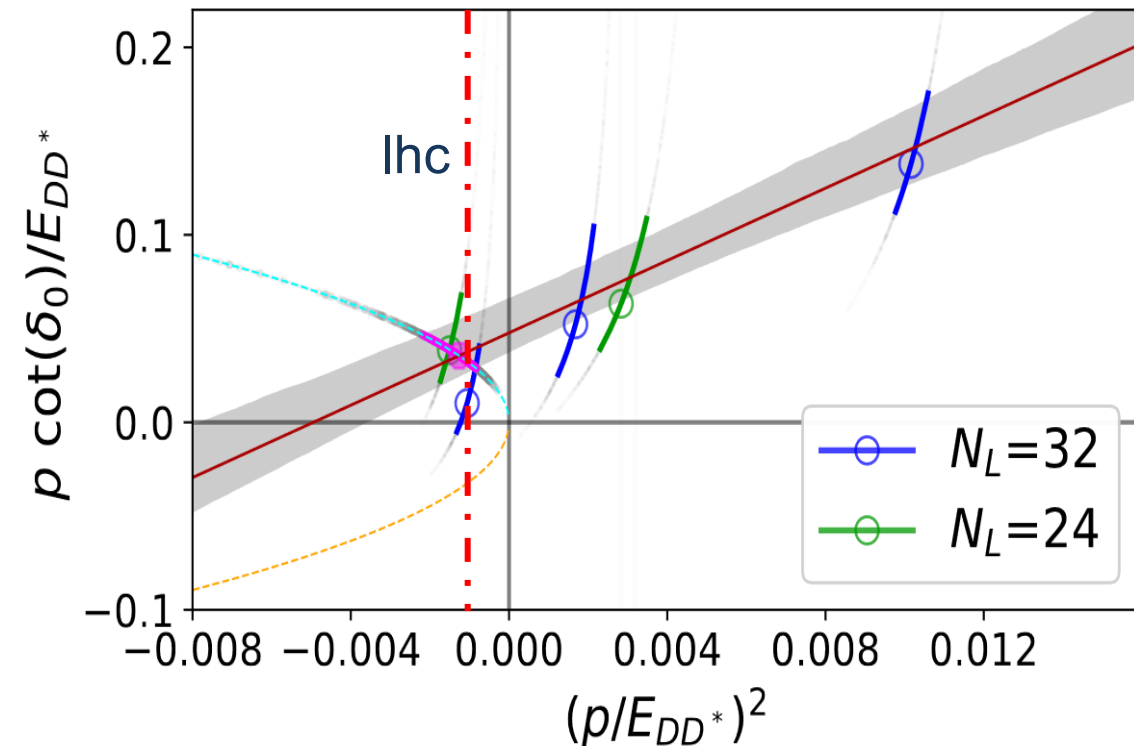
▶  $m_{\text{eff}}L = 2.6, 3.5$

▶  $p_{\text{lhc}}^2 \approx -\left(\frac{m_{\text{eff}}}{2}\right)^2 = -(126 \text{ MeV})^2$

▶  $p_{\text{rhc3}}^2 \approx 2\mu_{DD^*}(2m_D + m_\pi - m_D - m_{D^*}) \approx (560 \text{ MeV})^2$

- ①  $e^{-mL}$  effect can be neglected
- ② Considering the PW mixing effect
- ③  $E^{FV}$  well above lhc

④ ERE works in IFV Du:2023hlu



Other lattice results: Cheung:2017tnt, Junnarkar:2018twb, Chen:2022vpo, Lyu:2023xro

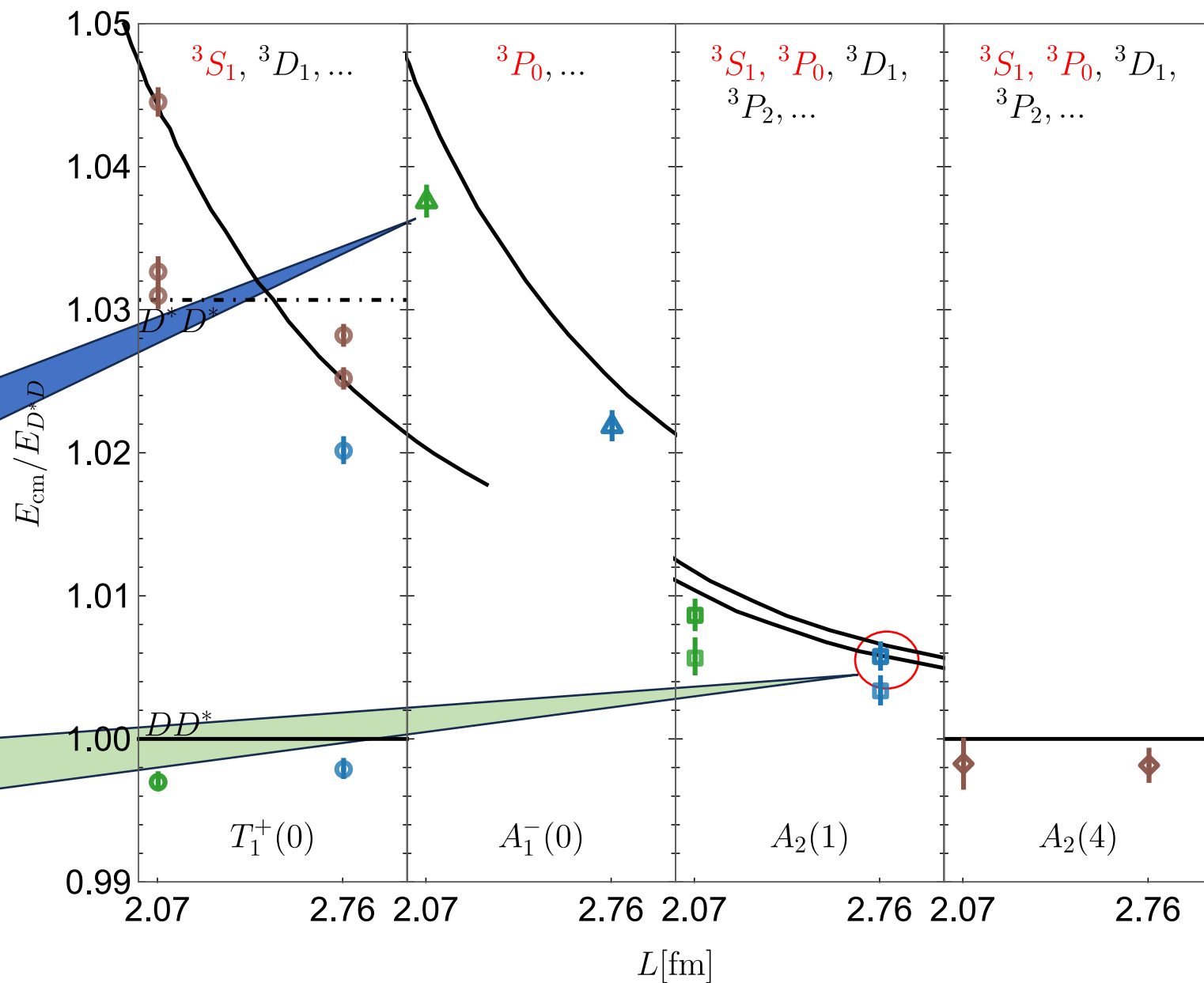
# Lattice FV energy levels

- Green and blue points as input
- 9 inputs in total

Highest input:  $p \approx 0.56$  GeV

- relativistic effect
- large cutoff  $\Lambda \geq 0.7$

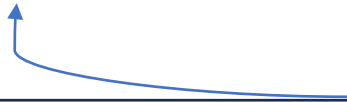
Overlap with noninteracting energy levels:  $\cot \theta = \infty$ ;  
Singular in  $p \cot \delta$  figures



- The energy levels of  $A_1^-(0)$  is high, relativistic formalism

$$T(\mathbf{p}, \mathbf{p}') = V(\mathbf{p}, \mathbf{p}') + \int \frac{d^3 \mathbf{q}}{(2\pi)^3} V(\mathbf{p}, \mathbf{q}) \frac{1}{2w_1 w_2} \frac{(w_1 + w_2)}{P_0^2 - (w_1 + w_2)^2 + i\epsilon} T(\mathbf{q}, \mathbf{p}')$$

$$w_i = \sqrt{m_i^2 + \mathbf{q}^2}$$



$$G = i \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(P - q)^2 - m_1^2 + i\epsilon} \frac{1}{q^2 - m_2^2 + i\epsilon}$$

- Replace integral into summation to get  $\mathbb{T} = \mathbb{V} + \mathcal{J} \mathbb{V} \cdot \mathbb{G} \cdot \mathbb{T}$

$$\int \frac{d^3 \mathbf{q}}{(2\pi)^3} \rightarrow \mathcal{J} \int \frac{d^3 \mathbf{q}_{box}}{(2\pi)^3} \rightarrow \mathcal{J} \sum_{\mathbf{n}} \frac{1}{L^3}$$

$\mathcal{J}$ : is the Jacobian determinant of the Lorentz boost

Li:2021mob

- Get the poles

$$\det(\mathbb{H} - \lambda \mathbb{I}) = 0 \rightarrow \mathbb{H} \mathbf{v} = \lambda \mathbf{v},$$

- Contact terms to NLO

$\mathcal{O}_1$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_4$	$\mathcal{O}_5$	$\mathcal{O}_6$
$\epsilon'^{\dagger} \cdot \epsilon$	$\mathbf{q}^2 \epsilon'^{\dagger} \cdot \epsilon$	$4\mathbf{k}^2 \epsilon'^{\dagger} \cdot \epsilon$	$4\mathbf{k} \cdot \epsilon'^{\dagger} \mathbf{k} \cdot \epsilon$	$\mathbf{q} \cdot \epsilon'^{\dagger} \mathbf{q} \cdot \epsilon$	$(\epsilon'^{\dagger} \times \epsilon) \cdot (\mathbf{q} \times 2\mathbf{k})$

$$\mathbf{q} = \mathbf{p}' - \mathbf{p},$$

$$\mathbf{k} = \frac{\mathbf{p}' + \mathbf{p}}{2}$$

$$\mathcal{O}_{^3S_1}^1 = \mathcal{O}_1$$

$$\mathcal{O}_{^3S_1}^2 = \mathcal{O}_2 + \mathcal{O}_3$$

$$\mathcal{O}_{^3S_1-^3D_1} = -\mathcal{O}_2 - \mathcal{O}_3 + 3\mathcal{O}_4 + 3\mathcal{O}_5$$

$$\mathcal{O}_{^3P_0} = -(\mathcal{O}_4 - \mathcal{O}_5) + \mathcal{O}_6$$

$$\mathcal{O}_{^3P_1} = -\frac{3}{2}(\mathcal{O}_2 - \mathcal{O}_3) + \frac{3}{2}(\mathcal{O}_4 - \mathcal{O}_5) + \frac{3}{2}\mathcal{O}_6$$

$$\mathcal{O}_{^3P_2} = -\frac{3}{2}(\mathcal{O}_2 - \mathcal{O}_3) - \frac{1}{2}(\mathcal{O}_4 - \mathcal{O}_5) - \frac{5}{2}\mathcal{O}_6$$

- In present calculation: LO and NLO  $^3S_1$  contact terms, NLO  $^3P_0$ 
  - ▶  $^3S_1$ -  $^3D_1$  transition term and  $^3P_2$  are included to estimate the systemic uncertainties

- Separable regulator:  $e^{-\frac{p^n+p'^n}{\Lambda^n}}$ ,  $n = 2,4,6$

# One-pion-exchange interaction

- Pion propagator: static approximation

$$D \approx -\mathbf{k}^2 - [m_\pi^2 - (M_{D^*} - M_D)^2] + i\epsilon$$

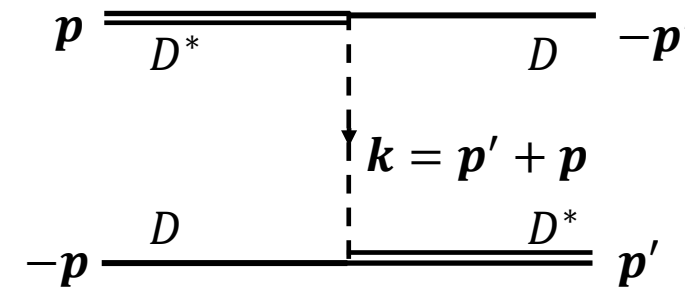
- Semilocal momentum-space regularization

$$\mathcal{V}(k) = -\frac{g^2}{4F_\pi} \left[ \frac{\mathbf{k} \cdot \boldsymbol{\epsilon}'^* \mathbf{k} \cdot \boldsymbol{\epsilon}}{k^2 + u^2} + C_{sub} \boldsymbol{\epsilon}'^* \cdot \boldsymbol{\epsilon} \right] e^{-\frac{k^2 + u^2}{\Lambda^2}}$$

$$C_{sub} = -\frac{\Lambda(\Lambda^2 - 2u^2) + 2\sqrt{\pi}u^3 e^{\frac{u^2}{\Lambda^2}} \text{erfc}(\frac{u}{\Lambda})}{3\Lambda^3}$$

- ▶ The regulator will not change the long-range behavior
- ▶ The short-range part of OPE is subtracted:  $V_{\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}}(r = 0) = 0$

Reinert:2017usi



# $F_\pi$ and $g$

- $F_\pi$  and  $g_{D^*D\pi}$  at  $m_\pi = 280$  MeV are determined by lattice QCD data, physical values by either linear extrapolation or chiral extrapolation

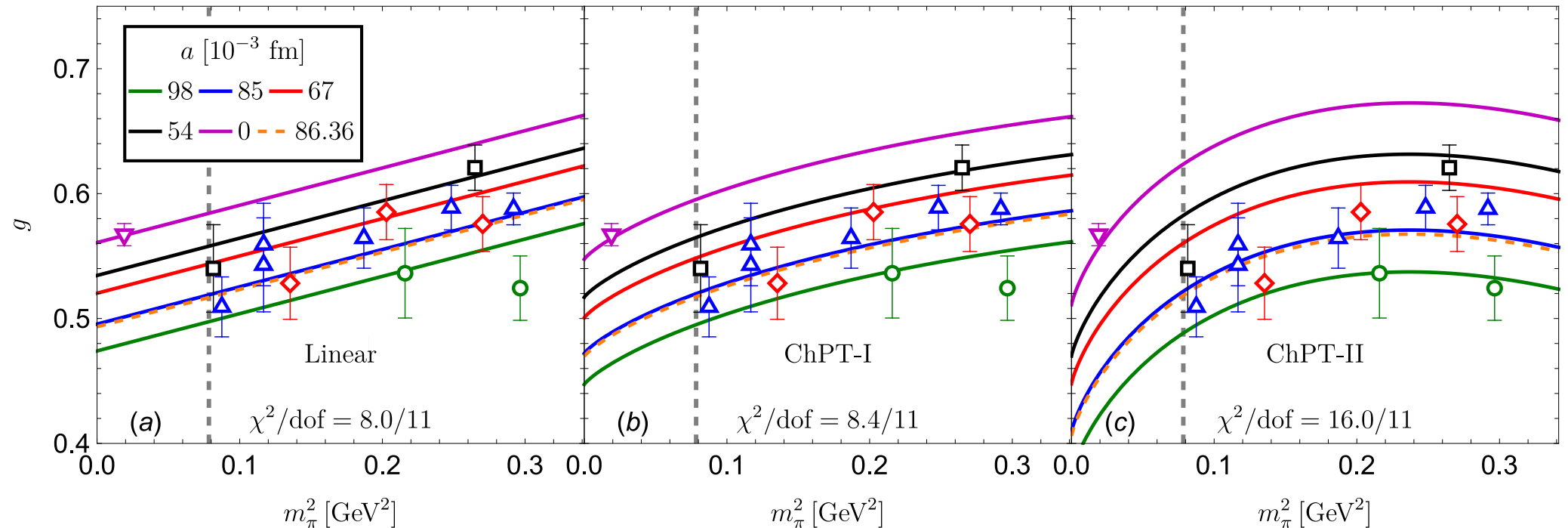
- $F_{ph} = 92.1$  MeV,  $F_0 = 85$  MeV, chiral extrapolation,  $\xi = m/m^{ph}$

$$f_\pi(\xi) = f_\pi^{ph} \left[ 1 + \left( 1 - \frac{f_0}{f_\pi^{ph}} \right) (\xi^2 - 1) - \frac{(m_\pi^{ph})^2}{8\pi^2 f_0^2} \xi^2 \log \xi \right]$$

Du:2023hlu, Becirevic:2012pf

- Three extrapolations give the consistent results

- ▶ The  $g$  is slightly smaller than the value in Ref. [Du:2023hlu]
- ▶  $g = 0.517 \pm 0.015$  for  $a = 0.086$  fm



# $\Lambda = 0.9 \text{ GeV}$ , only contact terms

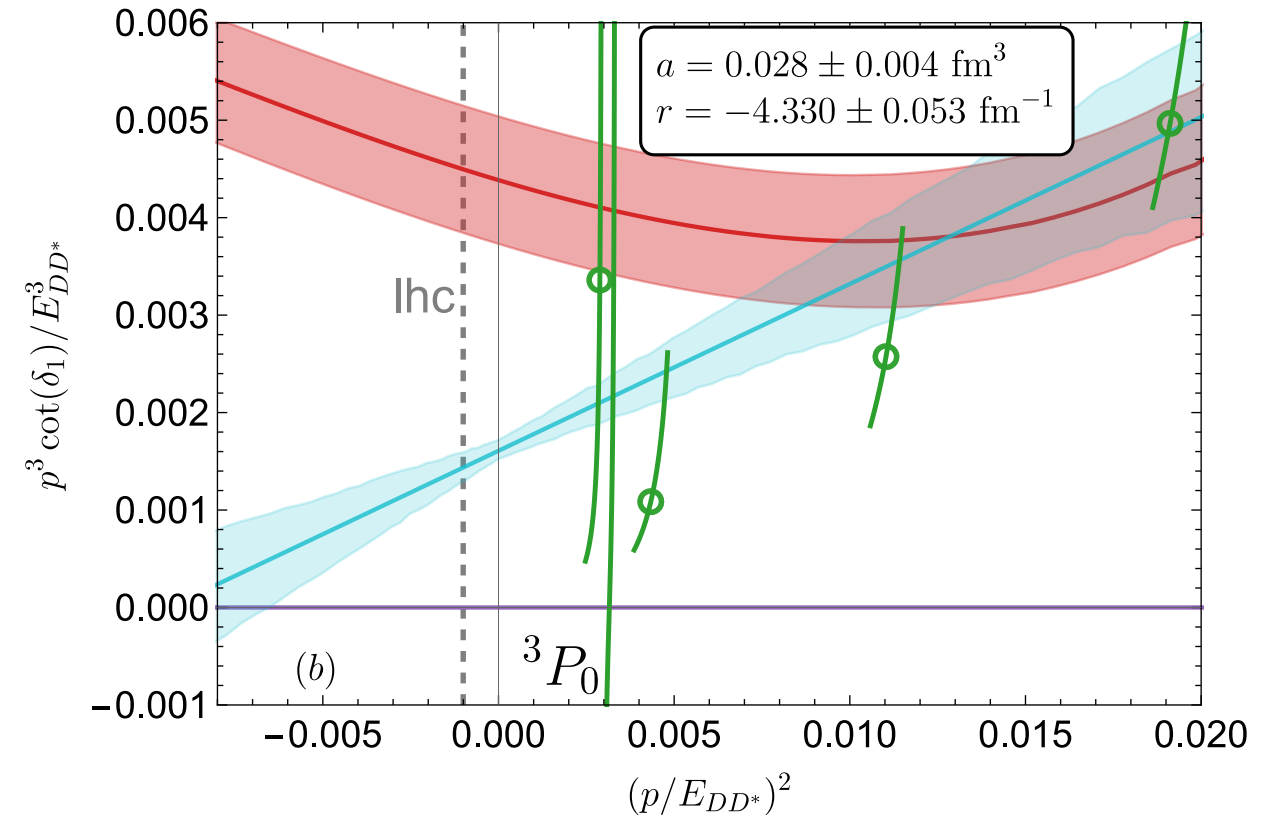
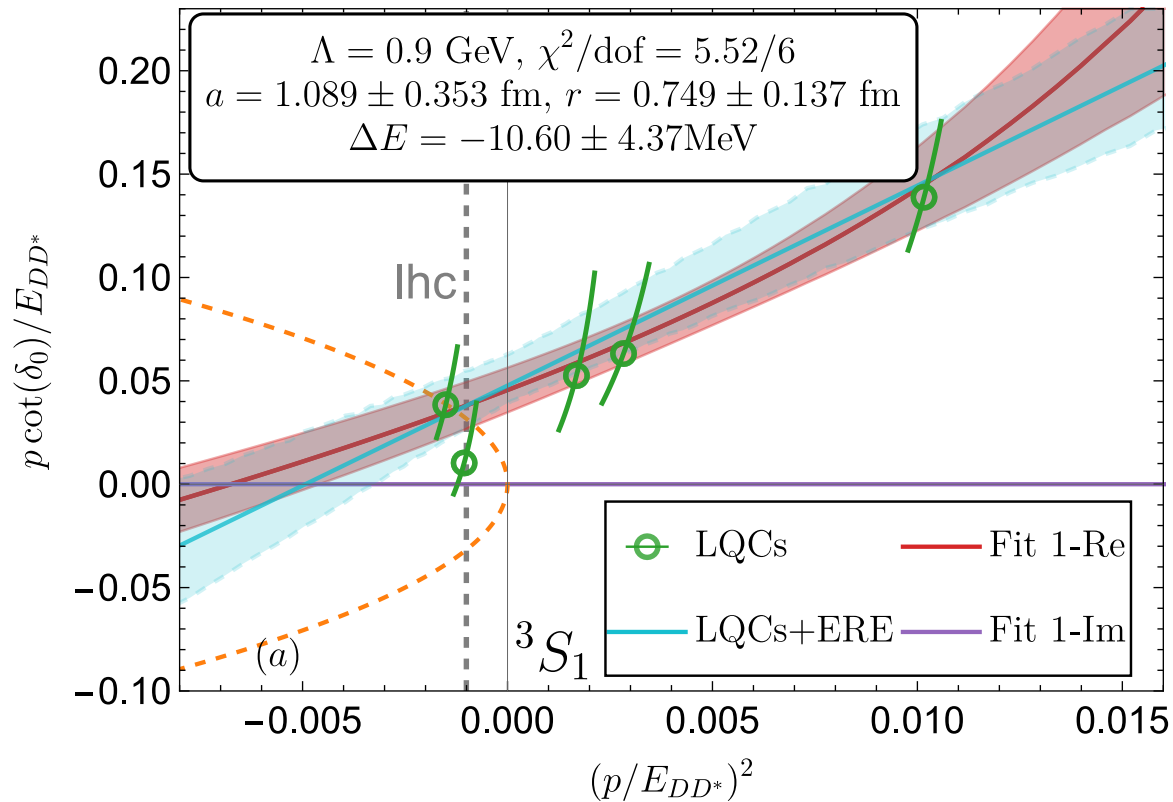
## ● Results using Lüscher's QCs Padmanath:2022cvi

$$\chi^2/\text{dof}=3.7/5, E_{\text{pole}}^{3S_1} = -9.9_{-7.2}^{+3.6} \text{ MeV}$$

$$a_{3S_1} = 1.04(29)\text{fm}, \quad r_{3S_1} = 0.96_{-0.20}^{+0.18}\text{fm}$$

$$a_{3P_0} = 0.076_{-0.009}^{+0.008}\text{fm}^3, \quad r_{3P_0} = 6.9(2.1)\text{fm}^{-1}$$

## ● Our results



# $\Lambda = 0.9 \text{ GeV}$ , contact terms+OPE

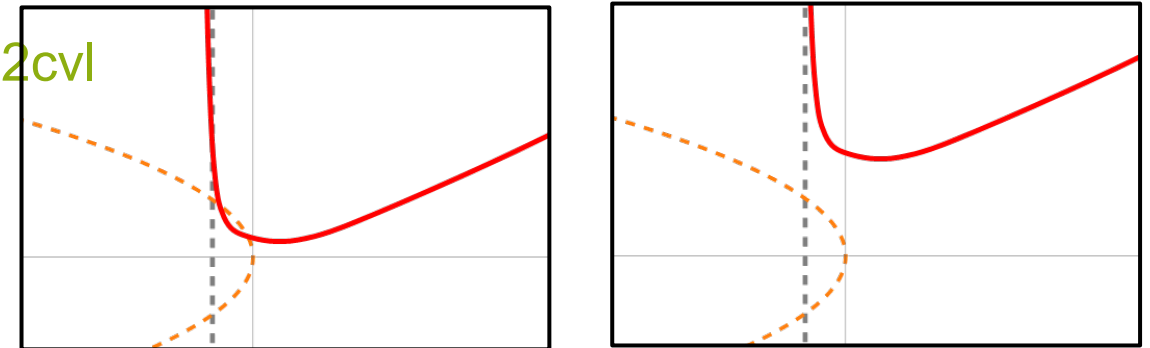
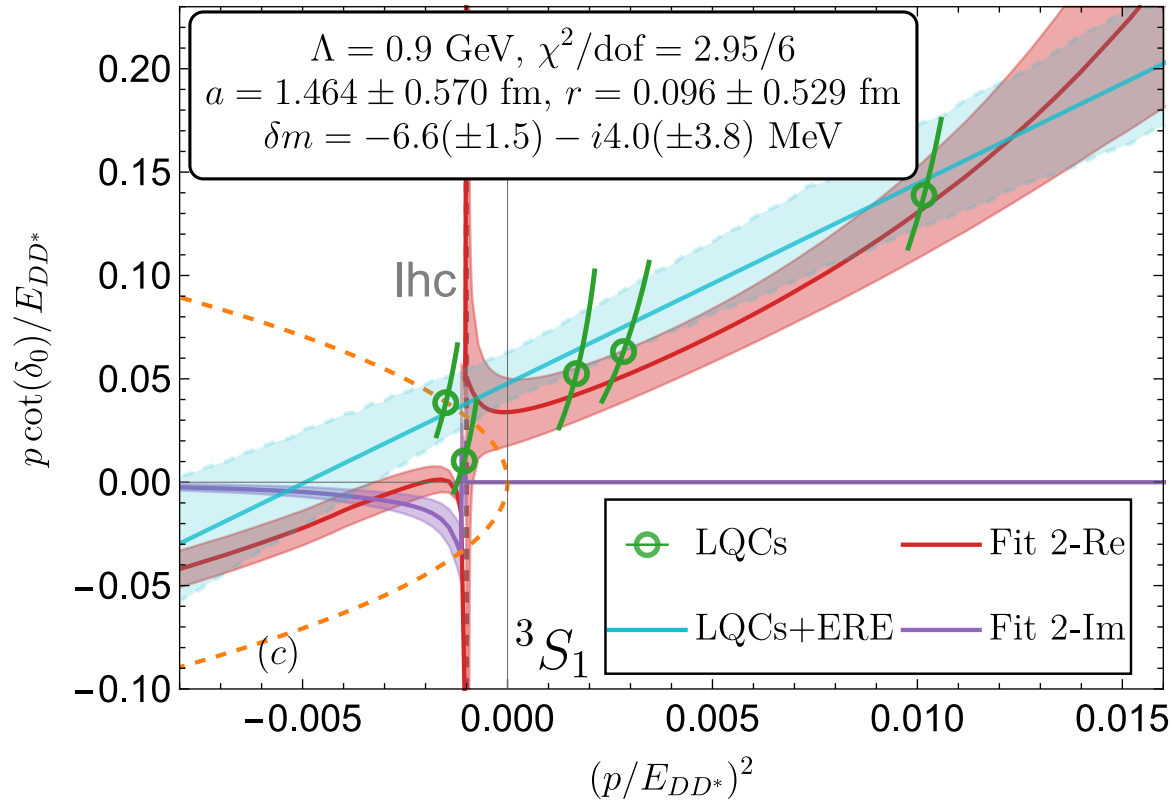
- Results using Lüscher's QCs Padmanath:2022cvi

$$\chi^2/\text{dof}=3.7/5, E_{\text{pole}}^{3S_1} = -9.9_{-7.2}^{+3.6} \text{ MeV}$$

$$a_{3S_1} = 1.04(29)\text{fm}, r_{3S_1} = 0.96_{-0.20}^{+0.18}\text{fm}$$

$$a_{3P_0} = 0.076_{-0.009}^{+0.008}\text{fm}^3, r_{3P_0} = 6.9(2.1)\text{fm}^{-1}$$

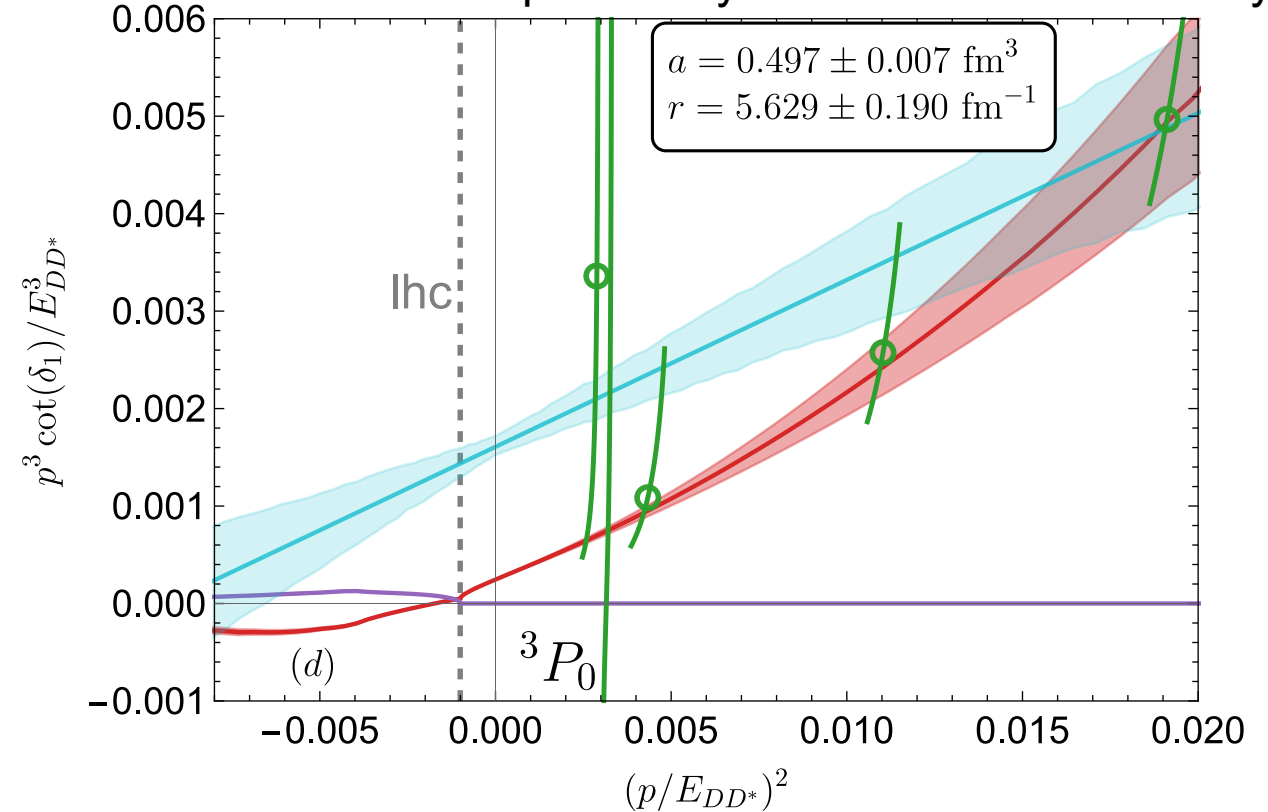
- Our results



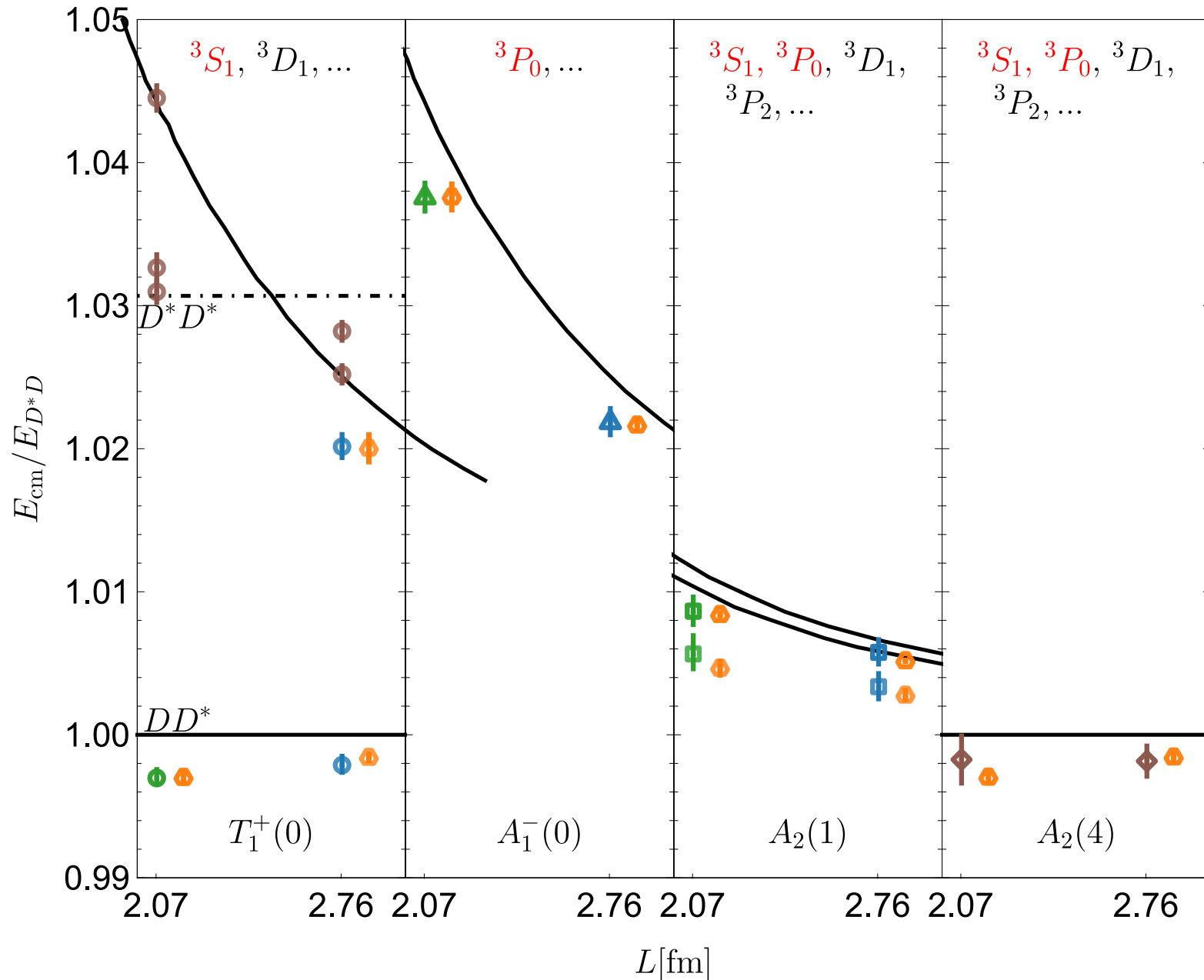
two virtual states

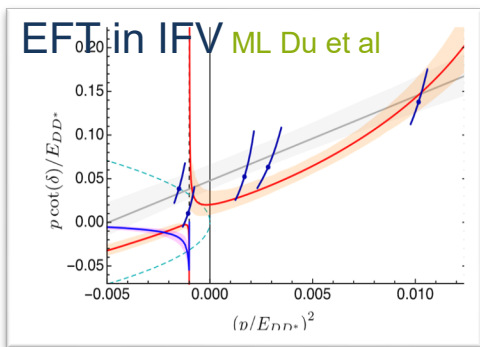
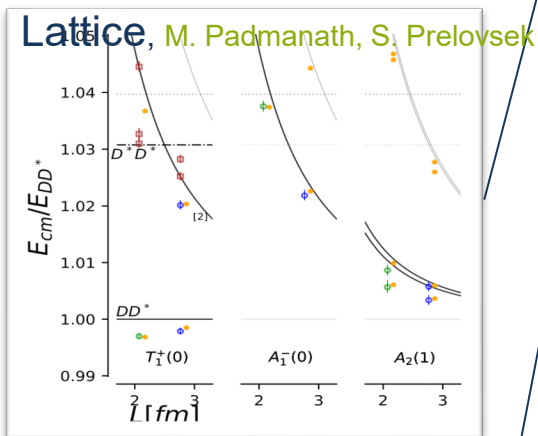
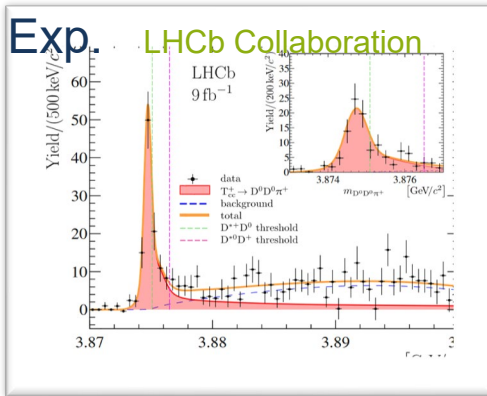
Resonance

Resonance with 85% probability within the  $1\sigma$  uncertainty



# Energy levels from the best fit





2108.02709

2109.01038

2202.10110

2303.09441

Lattice2023

2311.18793

2312.01930

**Summary Lattice2021**

- A proof-of-principle of an alternative method of Lüscher's formalism
- LSE or BSE in **plane wave** expansion+projection operator technique reduction to irreps
  - ⇒ Including partial wave mixing effect naturally, avoid complications of PW expansion
  - ⇒ Rest and moving two-particle systems, spinless, equal mass
- Non-relativistic example: spin-triplet NN
  - ⇒ S-wave dominant states: LF works well for  $L \gtrsim 5$  fm
  - ⇒ P-wave dominant states: OPE → large PW mixing effect regardless the box size
  - ⇒ EFT-based approach in the plane wave basis:  $V_{1\pi} + V_{1h} \xrightarrow[\text{data}]{\text{fit}} V_{1\pi} + V_{\text{cont}}^{(0)} + V_{\text{cont}}^{(2)} \dots$
  - ⇒ Advantages: 1) insensitive to PW mixing artifact; 2) small box (long-range interaction)

**EFT**

<b>Finite-volume scattering on the left-hand cut</b>	Andre Baião Raposo
Curia II, WH2SW	13:30 - 13:50
<b>Breakdown of Lüscher Formalism near Left Hand Cuts</b>	Md Habib E Islam
Curia II, WH2SW	13:50 - 14:10
<b>Resolving the left-hand-cut problem in lattice studies of the doubly-charmed tetraquark</b>	Steve Sharpe
Curia II, WH2SW	14:10 - 14:30

**Finite-volume scattering on the left-hand cut**

André Baião Raposo, Maxwell T. Hansen

Nov 30, 2023

59 pages

e-Print: [2311.18793 \[hep-lat\]](https://arxiv.org/abs/2311.18793)

**Modified Lüscher QCs framework and toy models**

**Solving the left-hand cut problem in lattice QCD:  $T_{cc}(3875)^+$  from finite volume energy levels**

Lu Meng, Vadim Baru, Evgeny Epelbaum, Arseniy A. Filin, Ashot M. Gasparyan

Dec 4, 2023

13 pages

e-Print: [2312.01930 \[hep-lat\]](https://arxiv.org/abs/2312.01930)

**EFT in FV**      **Application in Tcc**

# Two approaches

- Hansen's approach Raposo:2023oru

- ▶ FV: lattice data fix  $\bar{\mathcal{K}}^{OS}$

$$\det_{\mathbf{k}^* \ell m} \left[ S(P_j, L)^{-1} + \xi^\dagger \bar{\mathcal{K}}^{OS}(P_j) \xi + 2g^2 \mathcal{T}(P_j) \right] = 0$$

- ▶ IFV: solve a integral equation

$$\mathcal{M}^{\text{aux}}(P, p, p') = \mathcal{K}^\mathcal{T}(P, p, p') - \frac{1}{2} \int \frac{d^3 \mathbf{k}^*}{(2\pi)^3} \frac{\mathcal{M}^{\text{aux}}(P, p, k) H(\mathbf{k}^*) \mathcal{K}^\mathcal{T}(P, k, p')}{4\omega_N(\mathbf{k}^*) [(k_{OS}^*)^2 - (\mathbf{k}^*)^2 + i\epsilon]}$$

$$\mathcal{K}^\mathcal{T}(P, p, p') = \bar{\mathcal{K}}^{OS}(P, p, p') + 2g^2 \mathcal{T}(P, p, p'),$$

$$S_{\mathbf{k}^* \ell m, \mathbf{k}'^* \ell' m'}(P, L) = \frac{1}{2L^3} \frac{4\pi Y_{\ell m}(\hat{\mathbf{k}}^*) Y_{\ell' m'}^*(\hat{\mathbf{k}}^*) \delta_{\mathbf{k}^* \mathbf{k}'^*} |\mathbf{k}^*|^{\ell+\ell'} e^{-\alpha[(\mathbf{k}^*)^2 - (k_{OS}^*)^2]}}{4\omega_N(\mathbf{k}^*) [(k_{OS}^*)^2 - (\mathbf{k}^*)^2]},$$

$$\mathcal{T}_{\mathbf{k}^* \ell m, \mathbf{k}'^* \ell' m'}(P) = -\frac{1}{4\pi |\mathbf{k}^*|^\ell |\mathbf{k}'^*|^{\ell'}} \int d\Omega_{\hat{\mathbf{k}}^*} d\Omega_{\hat{\mathbf{k}}'^*} Y_{\ell m}(\hat{\mathbf{k}}^*) Y_{\ell' m'}^*(\hat{\mathbf{k}}'^*) \times \frac{1}{(p' - p)^2 - M_\pi^2}$$

$$\omega_N(\mathbf{p})^2 = m_N^2 + \mathbf{p}^2, \quad p = (\omega_N(\mathbf{k}^*), \mathbf{k}^*), \quad p' = (\omega_N(\mathbf{k}'^*), \mathbf{k}'^*)$$

$\xi = 1$  Model-independent?

You have to choose a parameterization of  $\bar{\mathcal{K}}^{OS}$ : ERE

To some how, the ERE is equivalent to the contact EFT

- Our approach

- ▶ FV: lattice data fix contact terms

$$\det[\mathbb{G}^{-1}(E) - \mathbb{V}] = 0.$$

- ▶ IFV: solve a integral equation

$$T(\mathbf{p}, \mathbf{p}', E) = V(\mathbf{p}, \mathbf{p}') + \int \frac{d^3 \mathbf{q}}{(2\pi)^3} V(\mathbf{p}, \mathbf{q}) G(\mathbf{q}, E) T(\mathbf{q}, \mathbf{p}', E)$$

$$V_{EFT} = \begin{array}{c} D^* \quad D \\ \text{---} \quad \text{---} \\ \quad \quad \quad \pi \\ \text{---} \quad \text{---} \\ D \quad D^* \end{array} + \begin{array}{c} D^* \quad D \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ D \quad D^* \end{array} + \dots$$

$$\mathbb{G}(E) = \frac{\mathcal{J}(\mathbf{q}_n)}{L^3} G(\mathbf{q}_n, E) \delta_{\mathbf{n}', \mathbf{n}}, \quad \mathbb{V} = V(\mathbf{q}_n, \mathbf{q}_{n'})$$

$$\begin{aligned} G(\mathbf{q}, E) &= i \int \frac{dq^0}{2\pi} \frac{1}{(P - q)^2 - m_1^2 + i\epsilon} \frac{1}{q^2 - m_2^2 + i\epsilon} \\ &= \frac{1}{4\omega_1 \omega_2} \left( \frac{1}{E - \omega_1 - \omega_2} - \frac{1}{E + \omega_1 + \omega_2} \right) \\ &= \frac{1}{2\omega_1 \omega_2} \frac{(\omega_1 + \omega_2)}{E^2 - (\omega_1 + \omega_2)^2 + i\epsilon} \end{aligned}$$

# Summary and Outlook

- Validation of Lüscher's formula

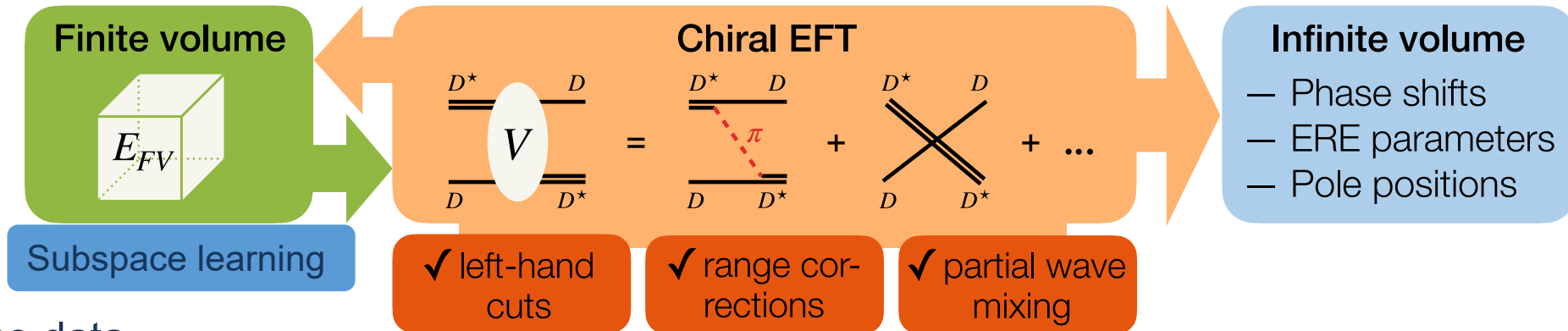
- ①  $e^{-mL}$  effect can be neglected
  - ② Considering the PW mixing effect
  - ③  $E^{FV}$  well above  $lhc$

④ ERE works in IFV Du:2023hlu

← Invalidate



- Our formalism



- $T_{cc}$  lattice data

- ▶ Better fit than QCs
- ▶ The possible partial wave mixing effect
- ▶  $E^{FV}$  below the left-hand cut

- Outlook

- ▶ 3-body system in the FV
- ▶ Identify OPE from QCD

One-pion exchange interaction

Yukawa, 1935

↓

Identify OPE from QCD, 2035?

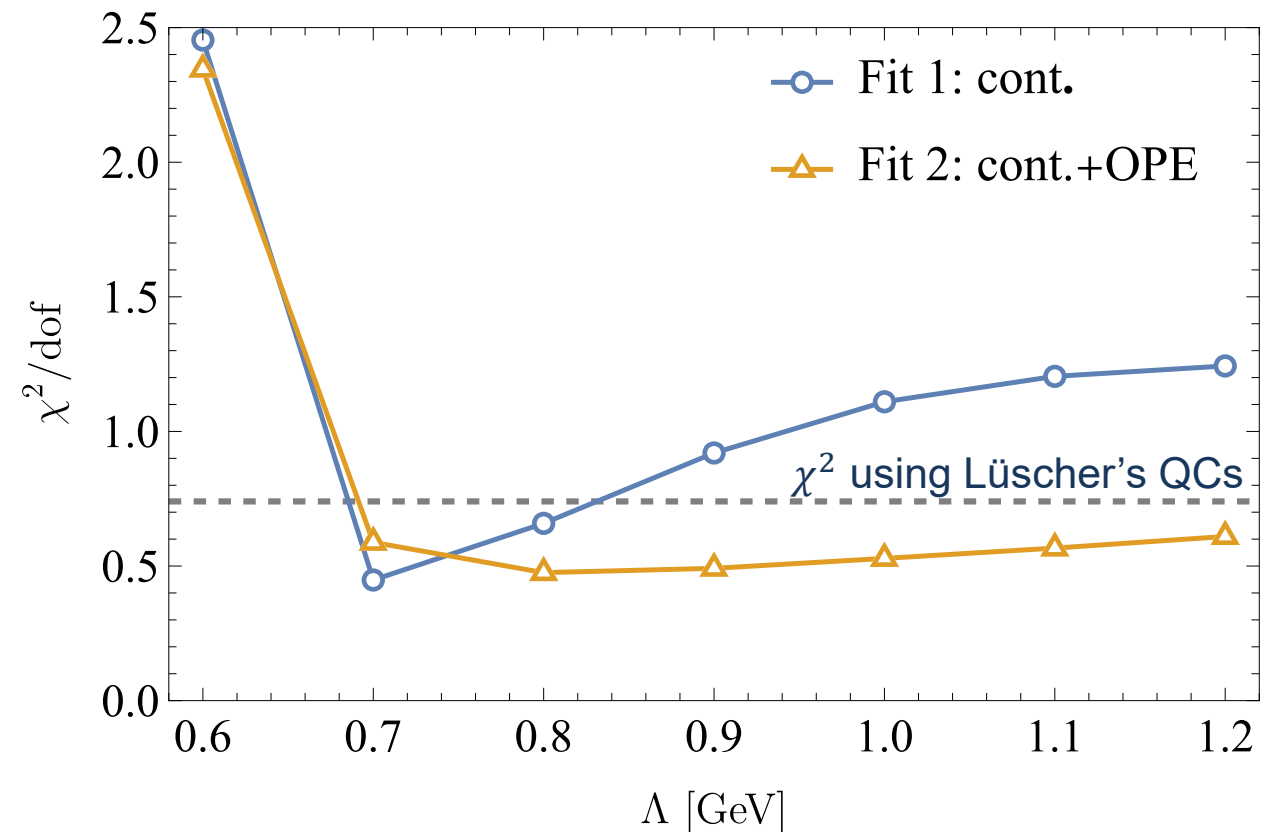
Thanks for your attentions!

---

**Back up**

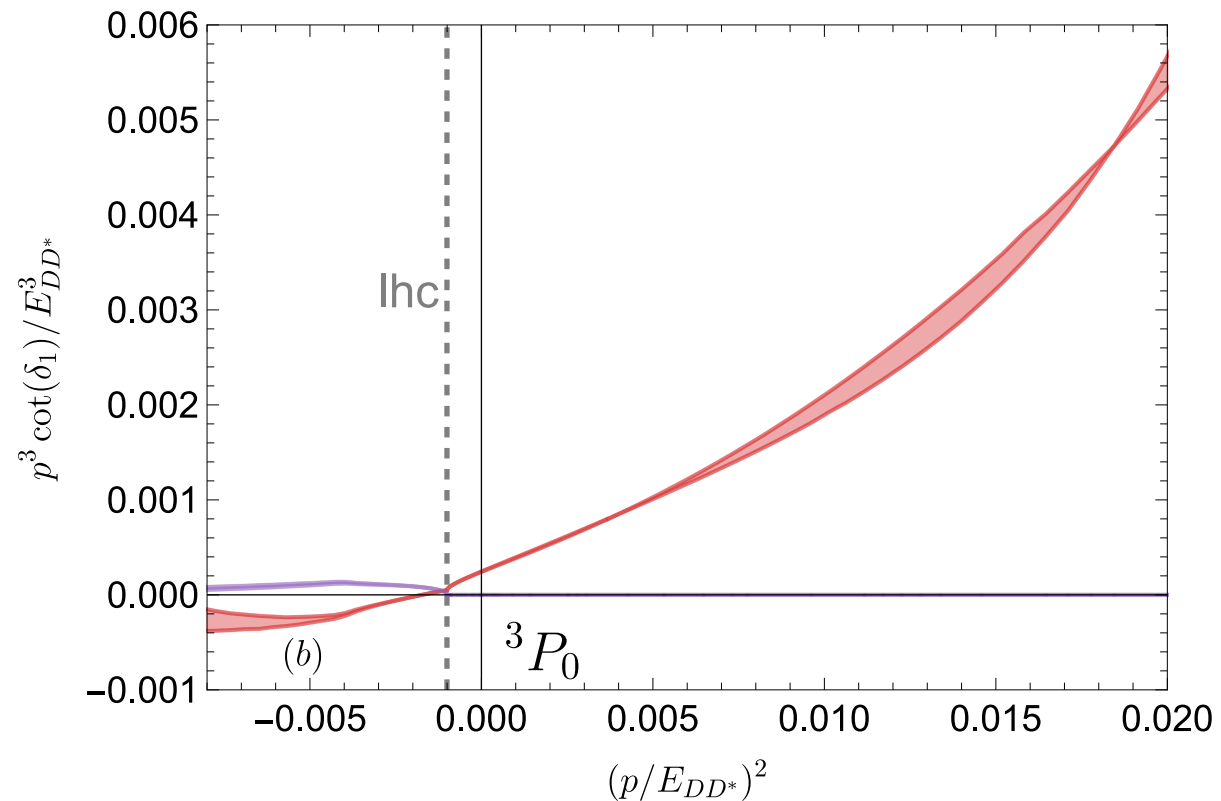
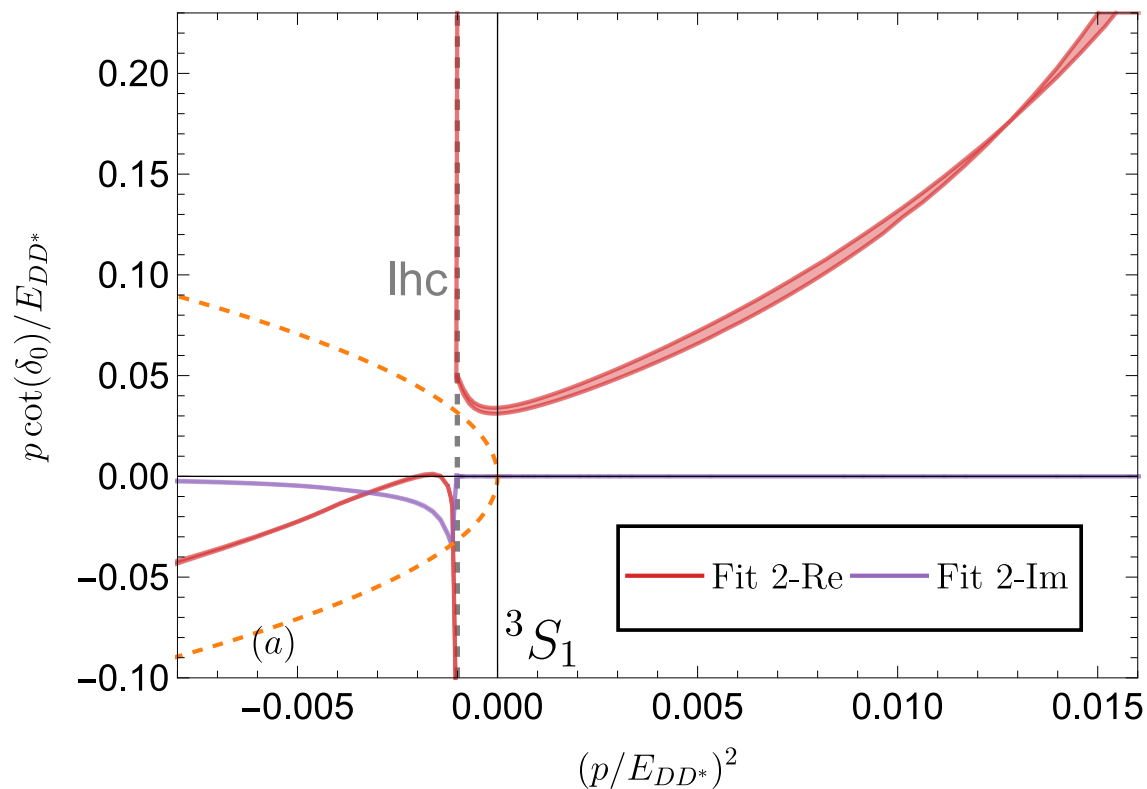
# Cutoff dependence of $\chi^2$

- 3 LECs: LO and NLO  $^3S_1$  contact terms, NLO  $^3P_0$
- In  $V_{\text{ctc}}$  fit, the P-wave dominate states control  $\Lambda$ -dependence of the  $\chi^2$ 
  - ▶ The shape of the of  $k^3 \cot \delta_1$  is determined by regulator and cutoff
  - ▶ Sensitive to  $\Lambda$
- The  $V_{\text{ctc}} + V_{1\pi}$  fit is stable with  $\Lambda$
- The  $V_{\text{ctc}} + V_{1\pi}$  fit is even better than QCs



# Cutoff dependence of phase shift

$\Lambda = 0.7 - 1.2$  GeV



# Detailed derivation of Lüscher's formula

$$C_L(P) = \text{O} \text{---} \Sigma \text{---} \text{O}^\dagger + \text{O} \text{---} \Sigma \text{---} V \text{---} \Sigma \text{---} \text{O}^\dagger + \dots$$

$$V = \text{X} + \text{---} \text{---} + \dots$$

$$\Sigma = \int + \Sigma - f \quad \Sigma - f \equiv F$$

$$C_L(P) = C_\infty(P) + A \text{---} F \text{---} A' + A \text{---} F \text{---} K \text{---} F \text{---} A' + \dots$$

$$A \text{---} = \text{O} \text{---} + \text{O} \text{---} \int \text{---} V \text{---} + \text{O} \text{---} \int \text{---} V \text{---} \int \text{---} V \text{---} + \dots$$

$$K = V + V \text{---} \int \text{---} V + \dots$$

Note: all the  $\int$  should be treated in the sense of P.V.

## • F cut

$$\left[ \frac{1}{L^3} \sum_k -\int \frac{d^3 \mathbf{k}}{(2\pi)^3} \right] f(k) = \begin{cases} \mathcal{O}(e^{-mL}) & \text{smooth } f(k) \\ \text{power of } L & \text{otherwise} \end{cases}$$

► Only the singularities are important

## • Within on-shell approximation:

$$G_F + G_F K G_F + \dots = G_F (1 - K G_F)^{-1} = (G_F^{-1} - K)$$

$$C_L(P) = C_\infty(P) + iA \frac{1}{G_F^{-1}(E, L) - K(P)} A^\dagger$$

## • Poles of $C_L$ is the FV energy levels

## • Lüscher's formula: $\det [G_F^{-1} - K] = 0$

Luscher:1990ux, Kim:2005gf, Polejaeva:2012ut

# Moving systems

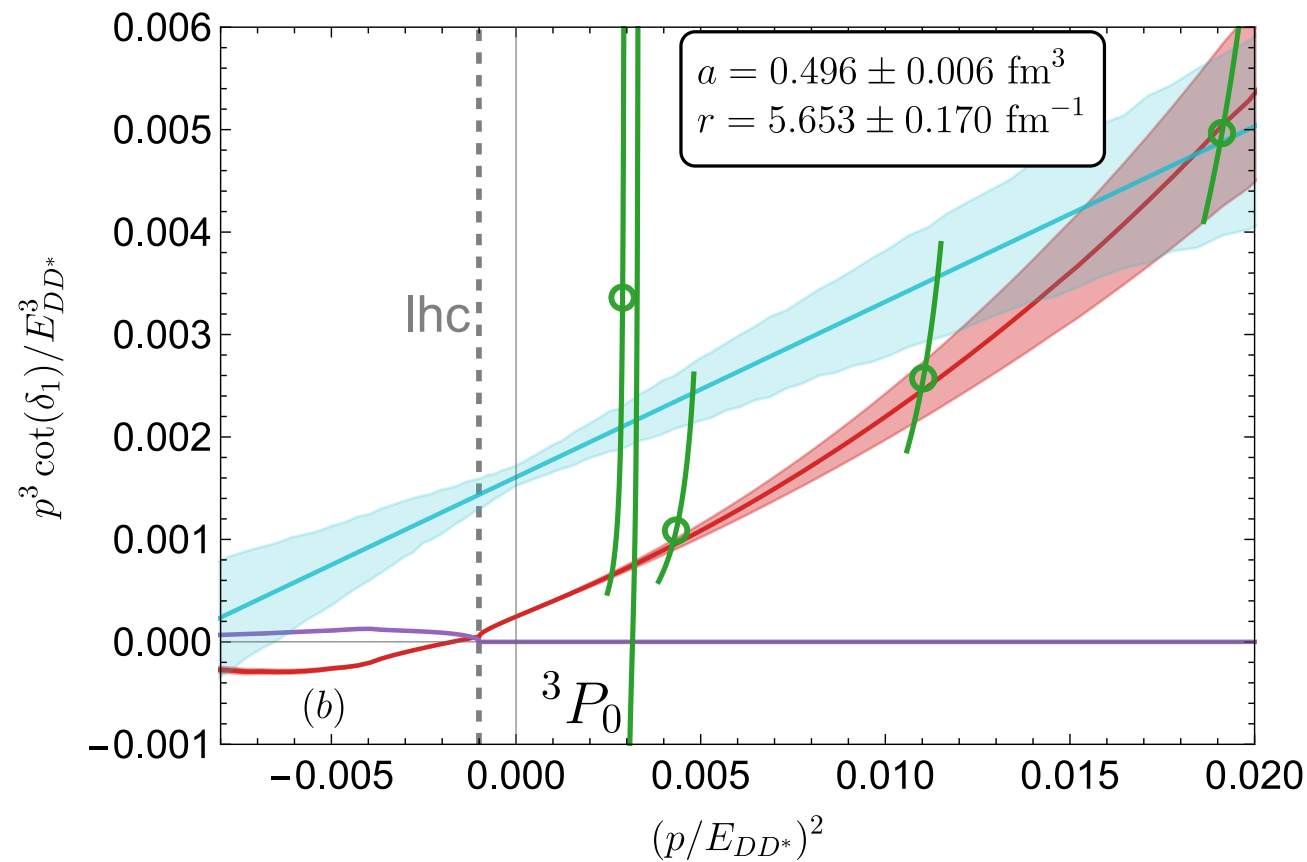
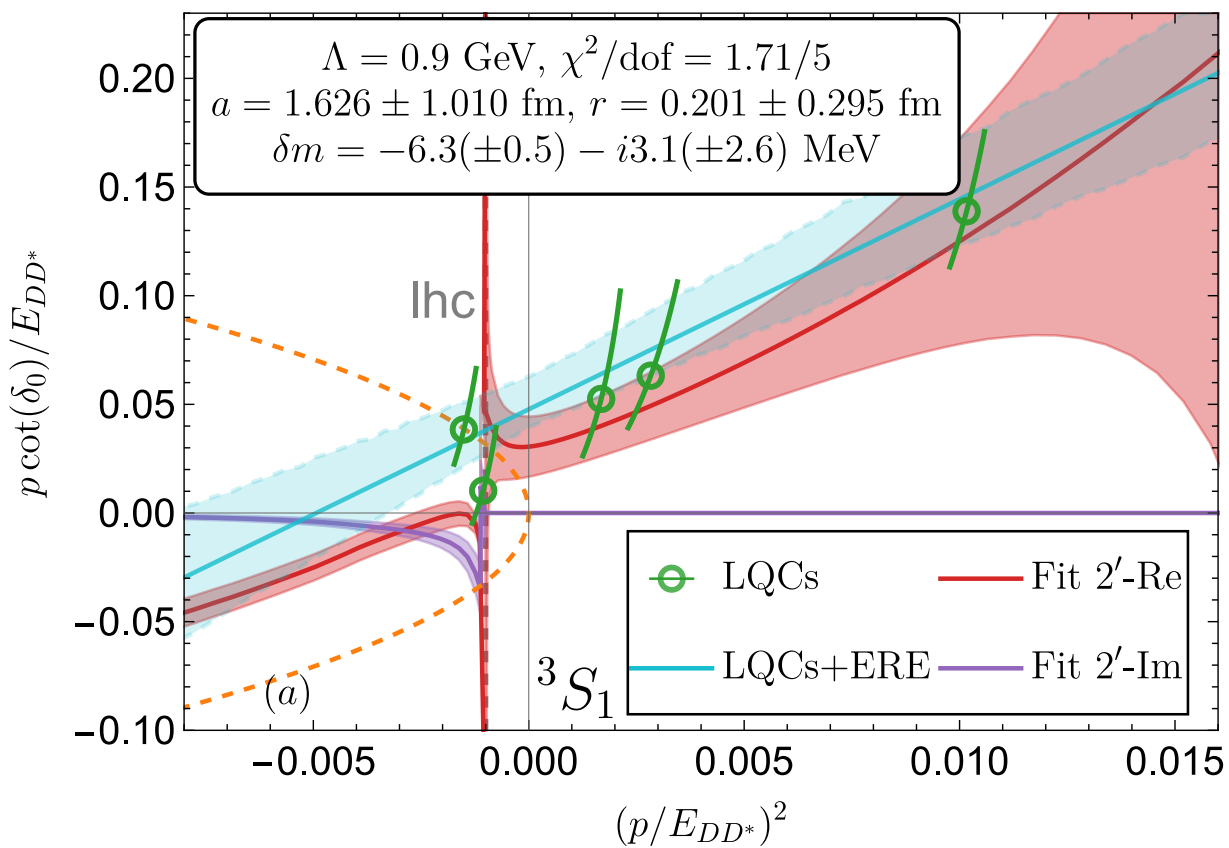
$m_1 = m_2, \quad A = 1$			$m_1 \neq m_2, \quad A = 1 + \frac{m_1^2 - m_2^2}{E^*}$		
$n \in Z$	$n - \frac{1}{2} \mathbf{d}$	$\gamma^{-1} \left( n_{\parallel} - \frac{\mathbf{d}}{2} \right) + n_{\perp}$	$n \in Z$	$n - \frac{A}{2} \mathbf{d}$	$\gamma^{-1} \left( n_{\parallel} - \frac{A}{2} \mathbf{d} \right) + n_{\perp}$
$\mathbf{d} = (0,0,1)$ 			$\mathbf{d} = (0,0,1)$ 		

Space inversion invariance is broken

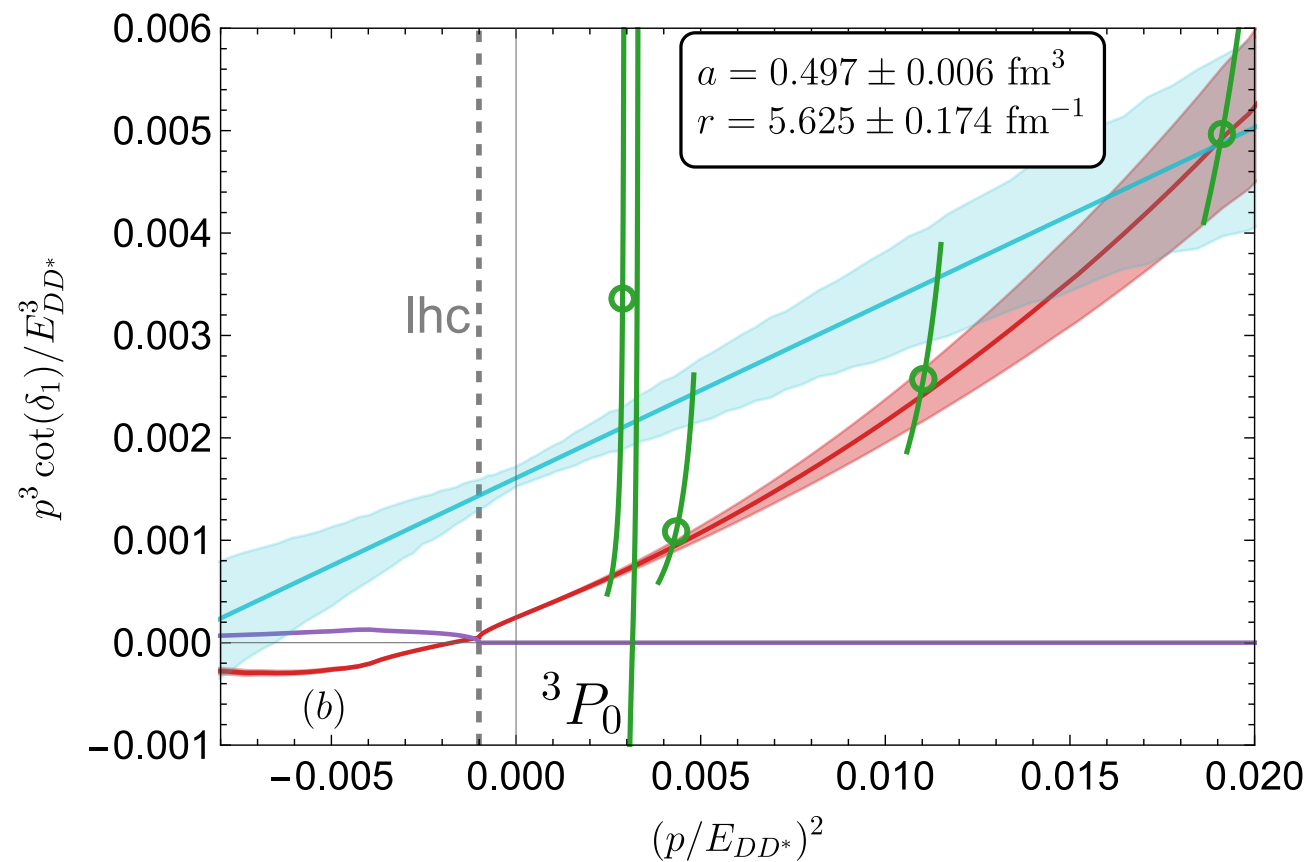
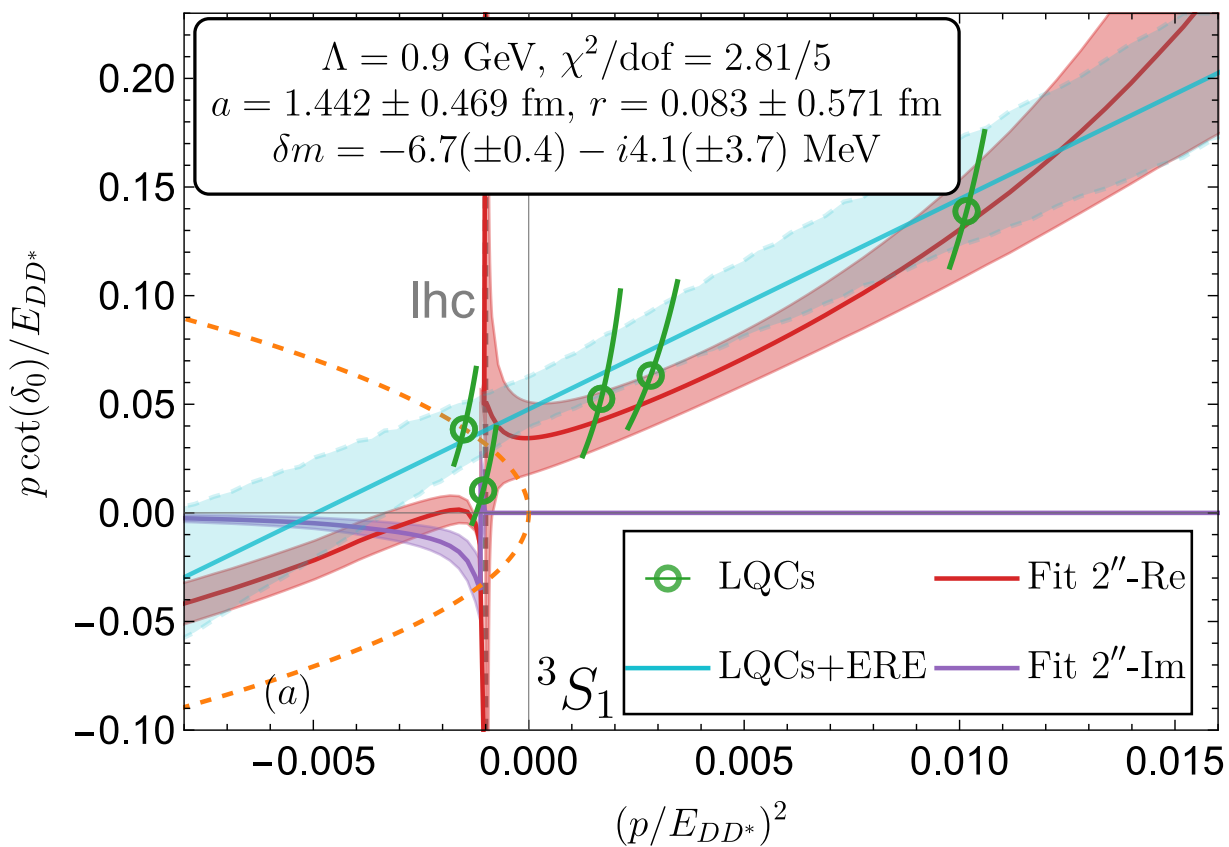
- Moving system in the box  $\mathbf{P} = \frac{2\pi}{L} \mathbf{d} \neq 0$ 
  - ▶ For LQCD, changing box size is expensive
  - ▶ Calculate  $E^{FV}$  of moving two-body systems in a box
- Box frame (BF)  $\mathbf{p}$  and center of mass frame (CMF)  $\mathbf{p}^*$ 
  - ▶ BF:  $\mathbf{p} = \frac{2\pi}{L} \mathbf{n}$ ; CMF:  $\mathbf{p}^* = \gamma^{-1} \left( \mathbf{p}_{\parallel} - \frac{A}{2} \mathbf{P} \right) + \mathbf{p}_{\perp}$
  - ▶ For moving systems with  $m_1 \neq m_2$ , states with different parities could mix
- $\mathbf{d} = (0,0,1)$ ,  $D_{4h}$  group for  $m_1 = m_2$ ,  $C_{4v}$  group for  $m_1 \neq m_2$
- $\mathbf{d} = (1,1,0)$ , ...

Rummukainen:1995vs,Leskovec:2012gb

# Including SD transition terms



# Including 3P2 term



- Lippmann-Schwinger equation in the finite volume

Luscher:1990ux,Polejaeva:2012ut

$$T^L(\mathbf{p}, \mathbf{q}; z) = V(\mathbf{p}, \mathbf{q}) + \int \frac{d^3 \mathbf{k}}{(2\pi)^3} V(\mathbf{p}, \mathbf{q}) G_0^L(\mathbf{k}; z) T(\mathbf{k}; z)$$

$$G_0^L(\mathbf{k}, z) = \left(\frac{2\pi}{L}\right)^3 \sum_{\mathbf{p} \in \frac{2\pi}{L} \mathbf{n}} \frac{2\mu \delta^3(\mathbf{p} - \mathbf{k})}{q_0^2 - \mathbf{p}^2} = \text{P.V.} \frac{2\mu}{q_0^2 - \mathbf{k}^2} + G_F(\mathbf{k}, z) = G_K(\mathbf{k}, z) + G_F(\mathbf{k}, z)$$

with  $z = m_1 + m_2 + \frac{q_0^2}{2\mu}$

- The “=” relation is valid up to the exponentially suppressed terms in  $L$
- $K$  matrix in the infinite volume:  $K = V + VG_K K$

$$T^L = V + V(G_K + G_F)T^L = K + KG_F T^L$$

- $E^{FV}$  corresponding to poles of  $T^L$  : interaction-independent form

$$\det[1 - KG_F] = 0, \text{ or } \det[G_F - K^{-1}] = 0$$

# Detailed derivation of Lüscher's formula

$$C_L(P) = \text{O} \text{---} \Sigma \text{---} \text{O}^\dagger + \text{O} \text{---} \Sigma \text{---} V \text{---} \Sigma \text{---} \text{O}^\dagger + \dots$$

$$V = \text{X} + \text{---} \text{---} + \dots$$

$$\Sigma = \int + \Sigma - \int$$

$$\Sigma - \int \equiv F$$

$$\left[ \frac{1}{L^3} \sum_k - \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \right] f(k) = \begin{cases} \mathcal{O}(e^{-mL}) & \text{smooth } f(k) \\ \text{power of } L & \text{otherwise} \end{cases}$$

$$C_L(P) = C_\infty(P) + A \text{---} F \text{---} A' + A \text{---} F \text{---} K \text{---} F \text{---} A' + \dots$$

Within on-shell approximation:

$$F + FKF + \dots = F(1 - KF)^{-1} = (F^{-1} - K)$$

$$A = \text{O} + \text{O} \text{---} \int \text{---} V + \text{O} \text{---} \int \text{---} V \text{---} \int \text{---} V + \dots$$

$$K = V + V \text{---} \int \text{---} V + \dots$$

Note: all the  $\int$  should be treated in the sense of P.V.

- Expanding it in partial wave basis

$$\det[G_F - K^{-1}] = 0, \Rightarrow \det[M_{l'm',lm} - \delta_{ll'}\delta_{mm'} \cot \delta_l] = 0$$

- ▶ Determinate equation of a matrix with infinite dimensions.
- ▶ Truncate at some  $l_{\max}$
- Reduce to irreps.  $\Gamma_i$  of point group

$$\det[F_{l'm',lm}] = 0 \Rightarrow \begin{vmatrix} F_{\Gamma_1} & & \\ & F_{\Gamma_2} & \\ & & \ddots \end{vmatrix} = 0, \Rightarrow \det[F_{\Gamma_i}] = 0$$

- Obtain the basis of the irreps.  $|l, m\rangle \rightarrow |\Gamma, l, \alpha\rangle$  (Projection operator technique)

Bernard:2008ax

- Lüscher quantization conditions:  $\det \left[ M_{ln,l'n'}^{(\Gamma, \mathbf{P})} - \delta_{ll'}\delta_{nn'} \cot \delta_l \right] = 0$

Luscher:1990ux, Rummukainen:1995vs, Feng:2004ua, Kim:2005gf, Fu:2011xz, Polejaeva:2012ut, Leskovec:2012gb, Gockeler:2012yj, ...

- Example  $\mathbf{d} = (0,0,1)$ ,  $\Gamma = A_1^+$ ,  $w_{lm}$  depends on  $E$  but independent on  $V$

$$\det \left[ M_{ln,l'n'}^{(\Gamma, \mathbf{P})} - \delta_{ll'} \delta_{nn'} \cot \delta_l \right] = 0, \quad M^{(A_1^+, \mathbf{d})} = \begin{bmatrix} w_{00} & -\sqrt{5}w_{20} & \cdots \\ -\sqrt{5}w_{20} & w_{00} + \frac{10}{7}w_{20} + \frac{18}{7}w_{40} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

- Truncate at  $l_{max} = 0$ , one-to-one relation:  $\delta_0(E^{FV}) \sim E^{FV}$
- Truncate at  $l_{max} > 0$ , no one-to-one relation
  - ▶ E.g.  $\{E_1^{FV}, E_2^{FV}\} \neq \{\delta_S(E_1^{FV}), \delta_S(E_2^{FV}), \delta_D(E_1^{FV}), \delta_D(E_2^{FV})\}$
  - ▶ One has to parameterize the K-matrix
  - ▶ Effective range expansion
- Lüscher's formula: quantization conditions in partial wave basis
- Why not quantization conditions in plane wave basis + Hamiltonian method?

# Hamiltonian approach in Plane wave basis: $|p_n, \eta\rangle$

- **Seven patterns** of representation space  $\{n_1, n_2, n_3\}_{dim}$  for  $O_h$  group

$$\Rightarrow \{0, 0, 0\}_{1 \times 3}, \{0, 0, a\}_{6 \times 3}, \{0, a, a\}_{12 \times 3}, \{0, a, b\}_{24 \times 3} \dots$$

- Reduce to irreducible representations (irreps): projection operator

e.g. textbook by M.Dresselhaus et.al

$$\hat{P}_{\alpha\beta}^{\Gamma_a} \equiv \sum_{g_i \in G} \frac{N(\Gamma_a)}{n_G} R_{\alpha\beta}^{\Gamma_a}(g_i)^* \hat{D}(g_i), \quad \hat{P}_{\alpha\alpha'}^{\Gamma_a} |\psi\rangle = a_{\alpha'}^{\Gamma_a} |\Gamma_a, \alpha\rangle.$$

- An example:  $\{0, 0, a\}_{6 \times 3} = 2T_1^+ \oplus T_2^+ \oplus A_1^- \oplus E_1^- \oplus T_1^- \oplus T_2^-$
- For moving systems, elongated boxes, particles with arbitrary spin...

Symmetric group (character table)  $\xrightarrow{\hat{P}^\Gamma}$  unitary irrep matrices  $\xrightarrow{\hat{P}_{\alpha\beta}^\Gamma}$  rep space  $|p_n\rangle \rightarrow$  irreps

- dim of the  $\mathbb{H}_\Gamma$  : cubic function of  $L^{-1}$

$$\dim \sim \left( \frac{\Lambda_{UV}}{2\pi/L} \right)^3 \times \frac{1}{10} \sim \mathcal{O}(1000)$$

# Test consistency with Lüscher's formula

- Using the LECs of our best fitting

- $V_{\text{full}}$  and S-wave-projected  $V_S$

- Obtain IFV T-matrix

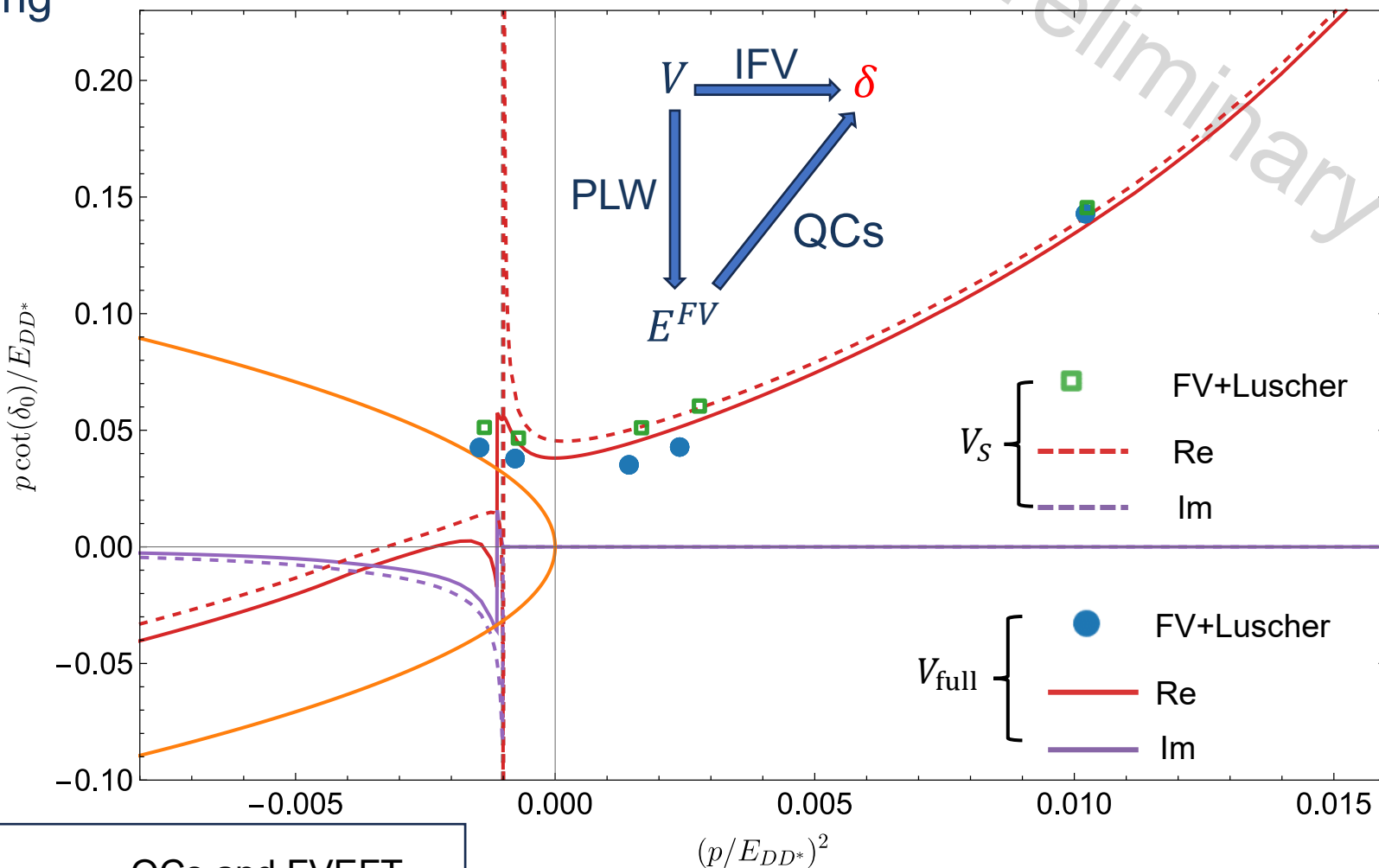
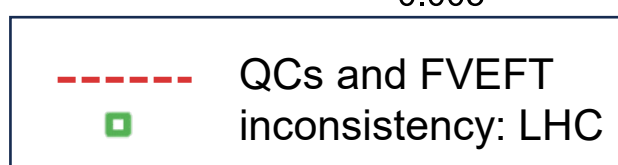
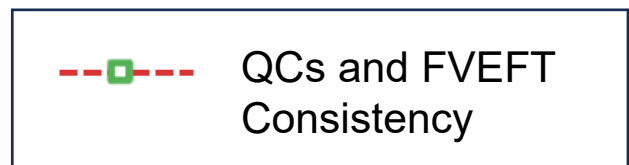
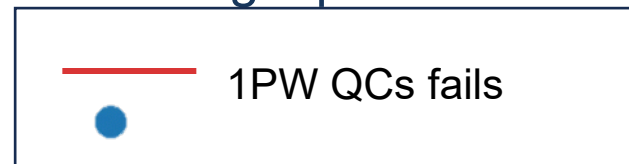


- Obtain the FV energy levels



- Using Lüscher's QCs to get  $\delta$

► Single partial wave QCs



Preliminary