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Left-hand cut problem in lattice QCD and an EFT-based solution

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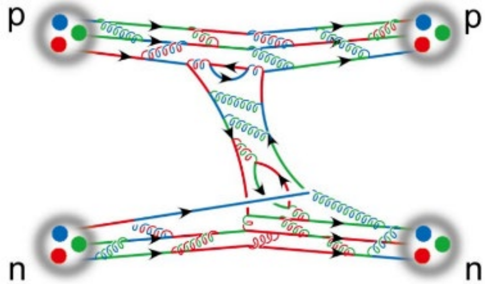
Base on JHEP10(2021)051, PoS LATTICE2022 (2023) 201 and arXiv:2312.01930

Together with V. Baru, E. Epelbaum, A. Filin, A.M. Gasparyan

Lattice QCD and finite volume energy levels

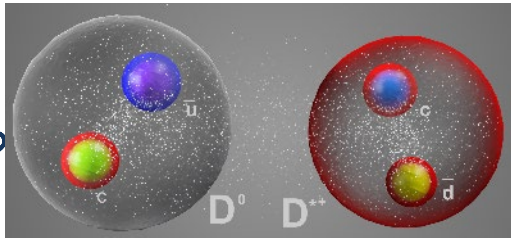
- QCD is the fundamental theory of the strong interaction

$$\mathcal{L}_{QCD} = \sum_f \bar{q}_f (i\not{D} - \mathcal{M} q_f) - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu,a}$$



Nuclear force

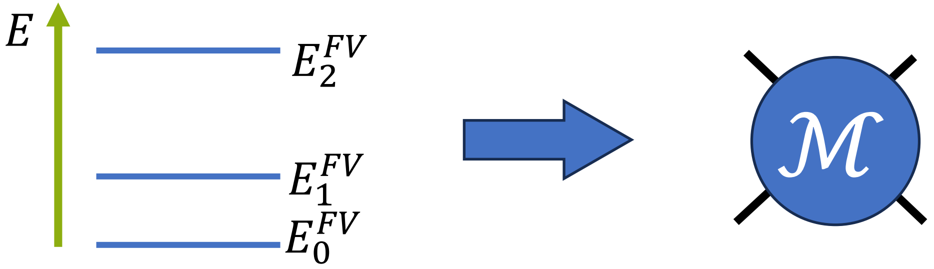
- Extract hadron-hadron interactions from QCD? Lattice QCD
  - ▶ formulated on a lattice of points in space and time in a finite volume (FV)



Hadronic molecules

- How to extract observables in the infinite volume (IFV) from a FV calculation?
  - ▶ Energy level method: Lüscher's formula, $E^{FV} \sim \delta_l(E^{FV})$

Luscher:1990ux

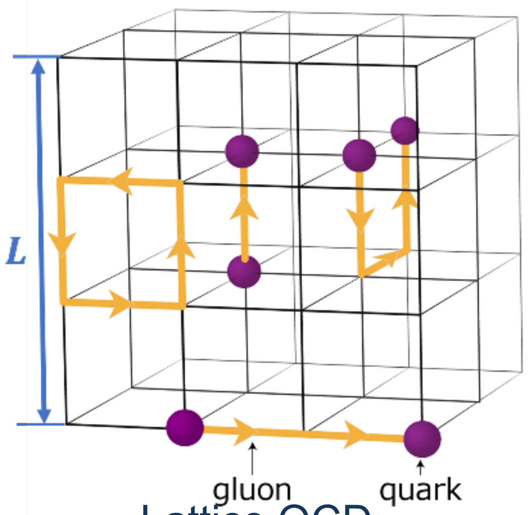


finite volume (FV) Infinite volume (IFV)

AKA : Lüscher Quantization conditions (LQCs)

  - ▶ Potential (or HAL QCD) method: Bethe–Salpeter amplitude → potential

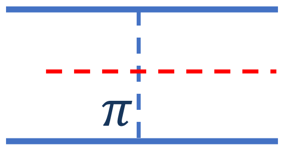
Ishii:2006ec



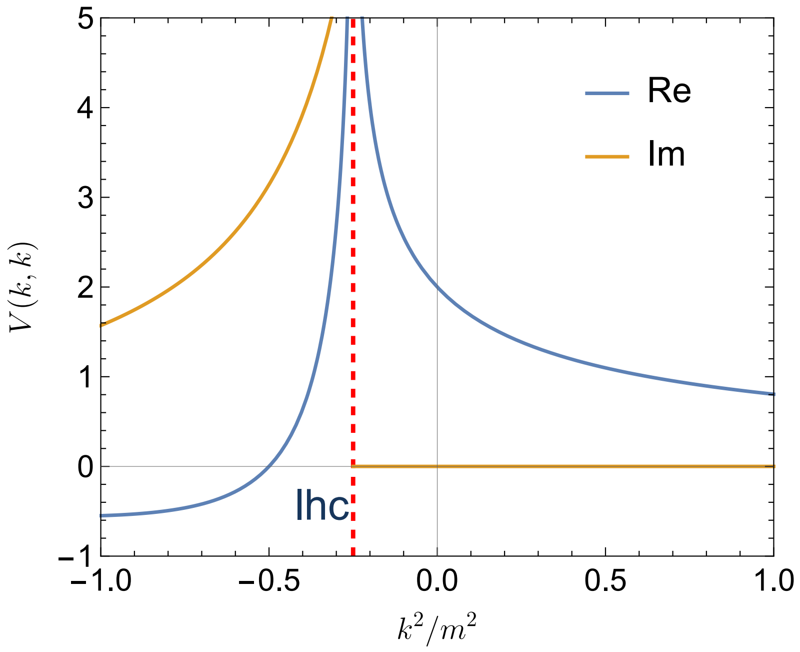
Lattice QCD

Left-hand cut

- Left-hand cut (lhc) from the one-pion exchange interaction



$$V(\vec{p},\vec{p}') = \frac{1}{(\vec{p}'-\vec{p})^2+m^2}$$



- Partial wave decomposition, e,g. S-wave

$$V_{l=0}(p,p') = \int_{-1}^{1} dz \frac{1}{p^2+p'^2-2pp'z+m^2} = -\frac{1}{2pp'}\log\left(\frac{(p-p')^2+m^2}{(p+p')^2+m^2}\right)$$

- On-shell $p = p' = k,\ \ k^2 = 2\mu E$

$$V_{l=0}(k,k) = \int_{-1}^{1} dz \frac{1}{2k^2(1-z)+m^2}$$

$$2k^2(1-z)+m^2 = 0 \Rightarrow z = \frac{m^2}{2k^2}+1 \qquad -1<z<1 \Rightarrow k^2 < -\frac{m^2}{4}$$

on-shell pion

▶ Branch point: $k^2 < -\frac{m^2}{4}$

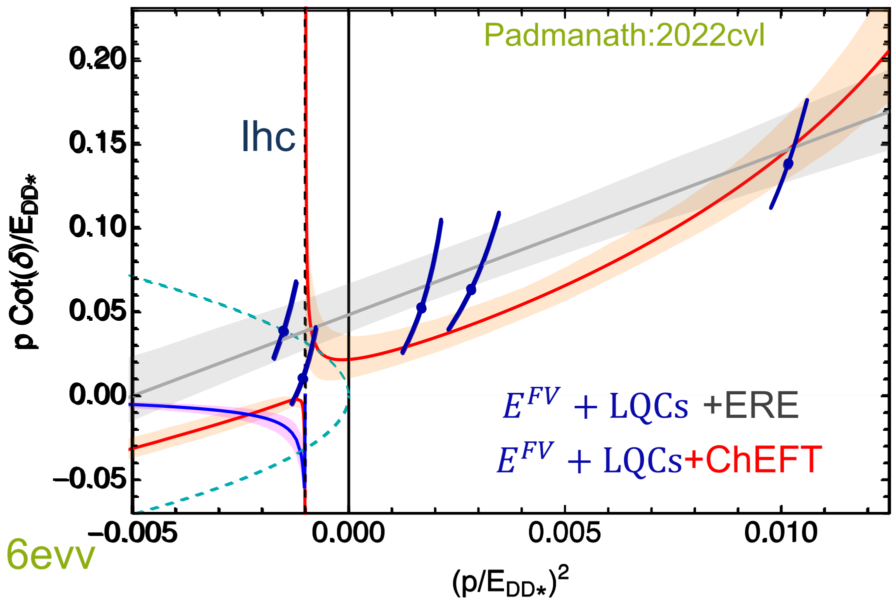
- K-matrix

$$K = V + VG^{\mathcal{P}}K$$

▶ The imaginary part arises

Left-hand problems for LQCs

- Lüscher formula: $\det\left[G_F^{-1}(L,E) - K(E)\right] = 0$
  kinematic term   K matrix in the IFV

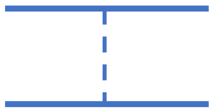


- Effective range expansion (ERE)

$$K^{-1}(p) = p\cot\delta(p) = \frac{1}{a} + \frac{1}{2}rp^2 + \cdots$$

  - Well used in lattice community
  - No left-hand cut

- NN system: modified ERE



Baru:2015ira, Baru:2016evv

Infinite volume

- DD^* at unphysical pion mass: Chiral effective field theory (ChEFT) Du:2023hlu
  - $K^{-1}(p) \neq \frac{1}{a} + \frac{1}{2}rp^2 + \cdots$
  - $V = V_{OPE} + V_{\mathrm{ctc}}^{LO} + V_{\mathrm{ctc}}^{NLO}$ and Lippmann-Schwinger eq. (LSE)

Rely on the δ_l from the Lüscher formula

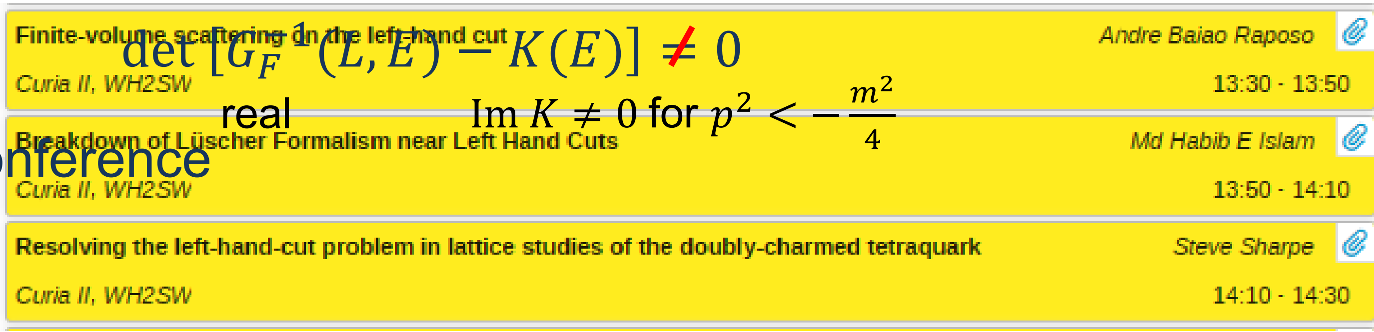
- The Lüscher formula fails because of the left-hand cut

$\det\left[G_F^{-1}(L,E) - K(E)\right] \neq 0$

real $\mathrm{Im}\,K \neq 0$ for $p^2 < -\frac{m^2}{4}$

  - Finite volume effect
  - Heated discussions in Lattice2023 conference

https://indico.fnal.gov/event/57249/



Finite-volume scattering on the left-hand cut Curia II, WH2SW	Andre Baiao Raposo	13:30 - 13:50
Breakdown of Lüscher Formalism near Left Hand Cuts Curia II, WH2SW	Md Habib E Islam	13:50 - 14:10
Resolving the left-hand-cut problem in lattice studies of the doubly-charmed tetraquark Curia II, WH2SW	Steve Sharpe	14:10 - 14:30

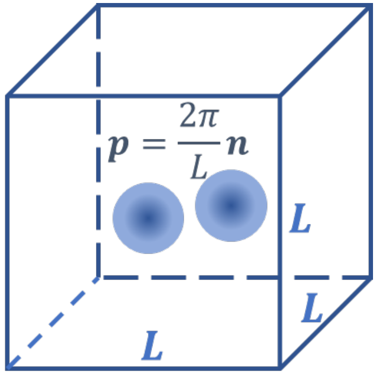
Contents

- Lüscher's formula
- Our formalism
- $T_{cc}(3875)^+$ state
- Summary and outlook

Lüscher's formula

Quantization of momentum

- Boundary conditions in the cubic box



$$\psi(\mathbf{x}_1, \mathbf{x}_2) = \psi(\mathbf{x}_1 + \mathbf{n}_1 L, \mathbf{x}_2 + \mathbf{n}_2 L)$$

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{P}, \quad \boldsymbol{p}_1 = \frac{2\pi}{L}\boldsymbol{n}, \quad \boldsymbol{P} = \frac{2\pi}{L}\boldsymbol{d}, \qquad \boldsymbol{n}, \boldsymbol{d} \in Z^3$$

  - ▶ 2-body rest systems: $\boldsymbol{d} = (0,0,0)$
- The rotation symmetry is broken: $SO(3) \to O_h$
  - ▶ $\{l, m\}$ are not good quantum numbers to label states
  - ▶ Partial wave mixing, for $l \neq l'$ and $m \neq m'$,

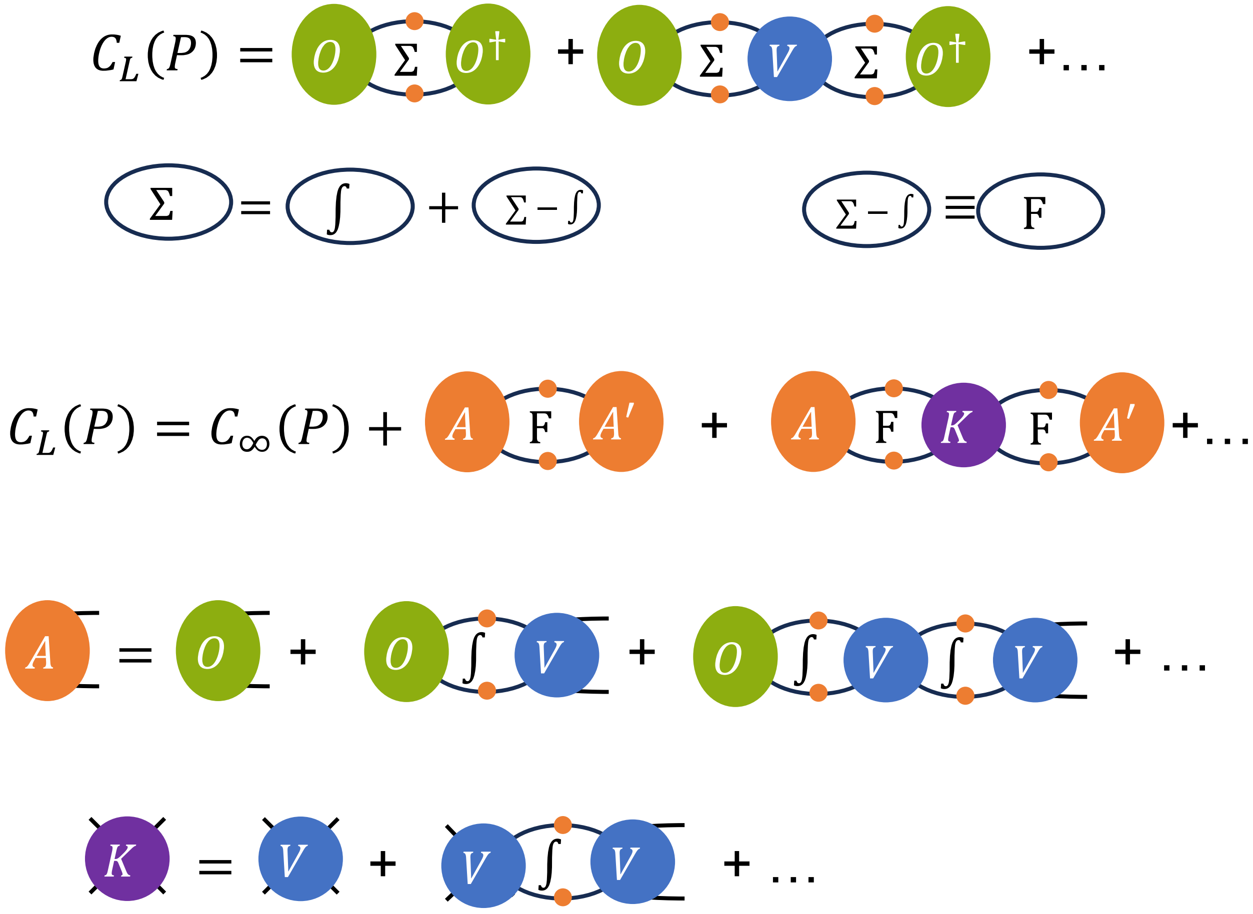
$$\langle lm | H^{FV} | l'm' \rangle \neq 0$$

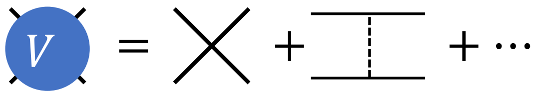
  - ▶ The FV energy should be classified by irreducible representations (irreps.) of O_h

$$\{l, m\} \to \{A_1, A_2, E, T_1, T_2\}$$

- Moving system in the box $\boldsymbol{P} = \frac{2\pi}{L}\boldsymbol{d} \neq 0$
  - ▶ Other point groups, D_{4h}, D_{2h} …
  - ▶ If $m_1 \neq m_2$, space inversion invariance is broken; states with different parities could mix

Detailed derivation of Lüscher's formula





- F cut

$$\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\right] f(k) = \begin{cases} \mathcal{O}(e^{-mL}) & \text{smooth } f(k) \\ \text{power of } L & \text{otherwise} \end{cases}$$

  - ▶ Only the singularities are important

- Within on-shell approximation:

$$G_F + G_F K G_F + \cdots = G_F(1 - K G_F)^{-1} = (G_F^{-1} - K)$$

$$C_L(P) = C_\infty(P) + iA \frac{1}{G_F^{-1}(E,L) - K(P)} A^\dagger$$

- Poles of C_L is the FV energy levels
- Lüscher's formula: det $[G_F^{-1} - K] = 0$

Note: all the ∫ should be treated in the sense of P.V.

Luscher:1990ux, Kim:2005gf, Polejaeva:2012ut

Lüscher's formula: partial wave mixing effect

- Expanding it in partial wave (PW) basis

$$\det[G_F - K^{-1}] = 0, \Rightarrow \det[M_{l'm',lm} - \delta_{ll'}\delta_{mm'}\cot\delta_l] = 0$$

  - ▸ Determinate equation of a matrix with infinite dimensions.
  - ▸ Truncate at some $l_{\max}$
- Reduce to irreps. Γ_i of point group:: $\det\left[M^{(\Gamma,\boldsymbol{P})}_{ln,l'n'} - \delta_{ll'}\delta_{nn'}\cot\delta_l\right] = 0$
- Example $\Gamma = A_1^+$, w_{lm} depends on E but independent on V

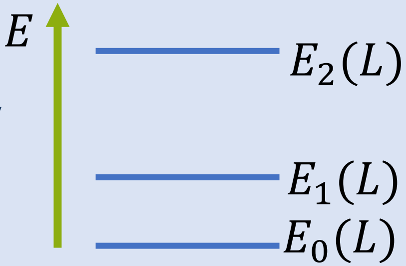
$$\det\left[M^{(\Gamma,\boldsymbol{P})}_{ln,l'n'} - \delta_{ll'}\delta_{nn'}\cot\delta_l\right] = 0, \quad M^{(A_1^+,\boldsymbol{d})} = \begin{bmatrix} w_{00} & -\sqrt{5}w_{20} & \cdots \\ -\sqrt{5}w_{20} & w_{00} + \frac{10}{7}w_{20} + \frac{18}{7}w_{40} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Bernard:2008ax

- Truncate at $l_{max} = 0$, one-to-one relation: $\delta_0(E^{FV}) \sim E^{FV}$
- Truncate at $l_{max} > 0$, no one-to-one relation
  - ▸ E.g. $\{E_1^{FV}, E_2^{FV}\} \not\Rightarrow \{\delta_S(E_1^{FV}), \delta_S(E_2^{FV}), \delta_D(E_1^{FV}), \delta_D(E_2^{FV}) \ldots\}$
  - ▸ One has to parameterize the K-matrix: e.g. effective range expansions (ERE)

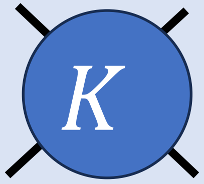
Luscher:1990ux,Rummukainen:1995vs,Feng:2004ua,Kim:2005gf,Fu:2011xz,Polejaeva:2012ut,Leskovec:2012gb,Gockeler:2012yj,...

FV energy spectrum



$\det\left[G_F^{-1} - K\right] = 0$

① e^{-mL} effect can be neglected
② Considering the PW mixing effect
③ E^{FV} well above lhc

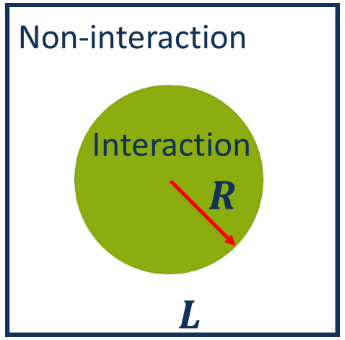


IFV K-matrix

④ERE works

Du:2023hlu

$$k\cot\delta(k) = \frac{1}{a} + \frac{1}{2}rk^2 + \cdots$$



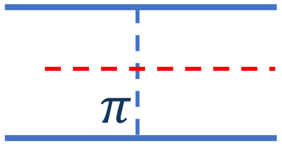
Requirement: $\frac{L}{2} \gg R$

Typically: $m_\pi L > 3$

High partial wave suppression

- Threshold effect: $T_l(p) \sim p^{2l}$
- The large scale: m_π

Taylor's textbook P197



Left-hand cut

All four requirements constrained by the $V_{1\pi}$

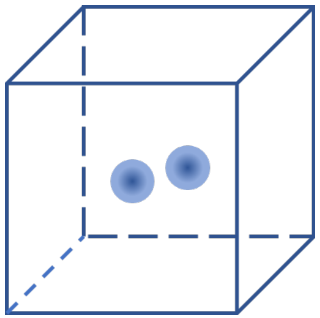
Our strategy

Solving the lhc problem

- Our strategy: solve the Schrödinger Eq. to get the **bound state** solutions

$$\frac{\boldsymbol{p}^2}{2\mu}\psi(\boldsymbol{p}) + \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} V(\boldsymbol{p},\boldsymbol{p}')\psi(\boldsymbol{p}) = E\psi(\boldsymbol{p})$$

Off-shell, for $\mathrm{E} < 0$

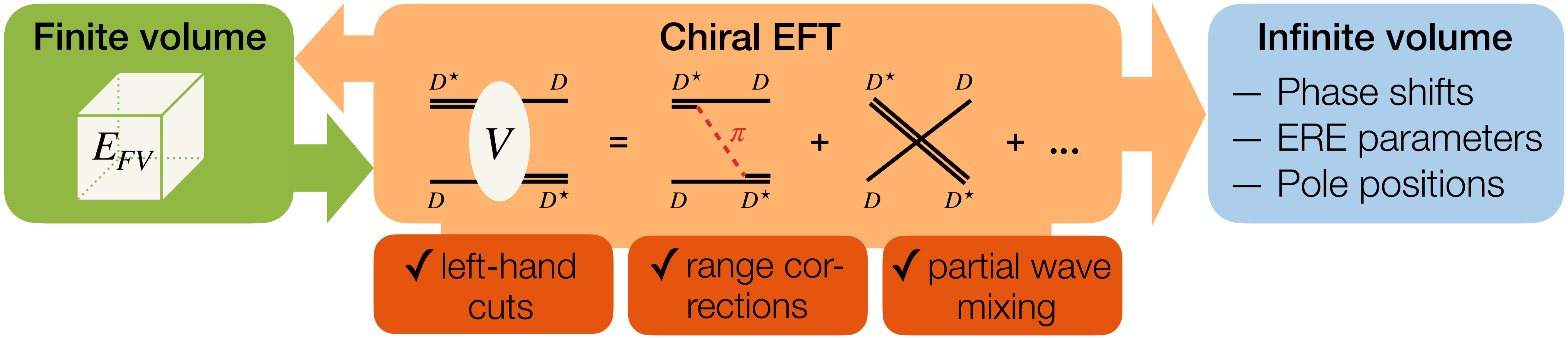


- ▶ Works well even for $2\mu E < -\frac{m^2}{4}$
- ▶ In the finite volume:

$$\int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \to \frac{1}{L^3}\sum_{\boldsymbol{p}_n}$$

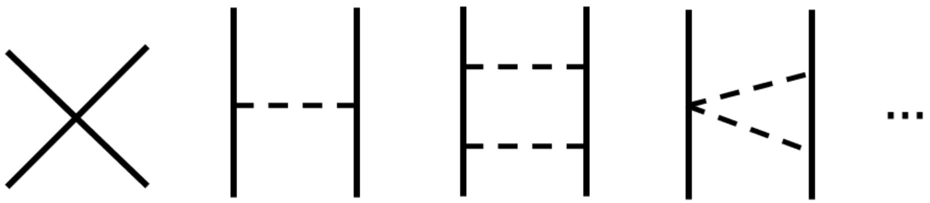
E_{FV} are "bound states" trapped by the potential well

- ▶ Include the OPE explicitly
- ▶ Plane wave basis with discrete momentum: cubic group symmetry



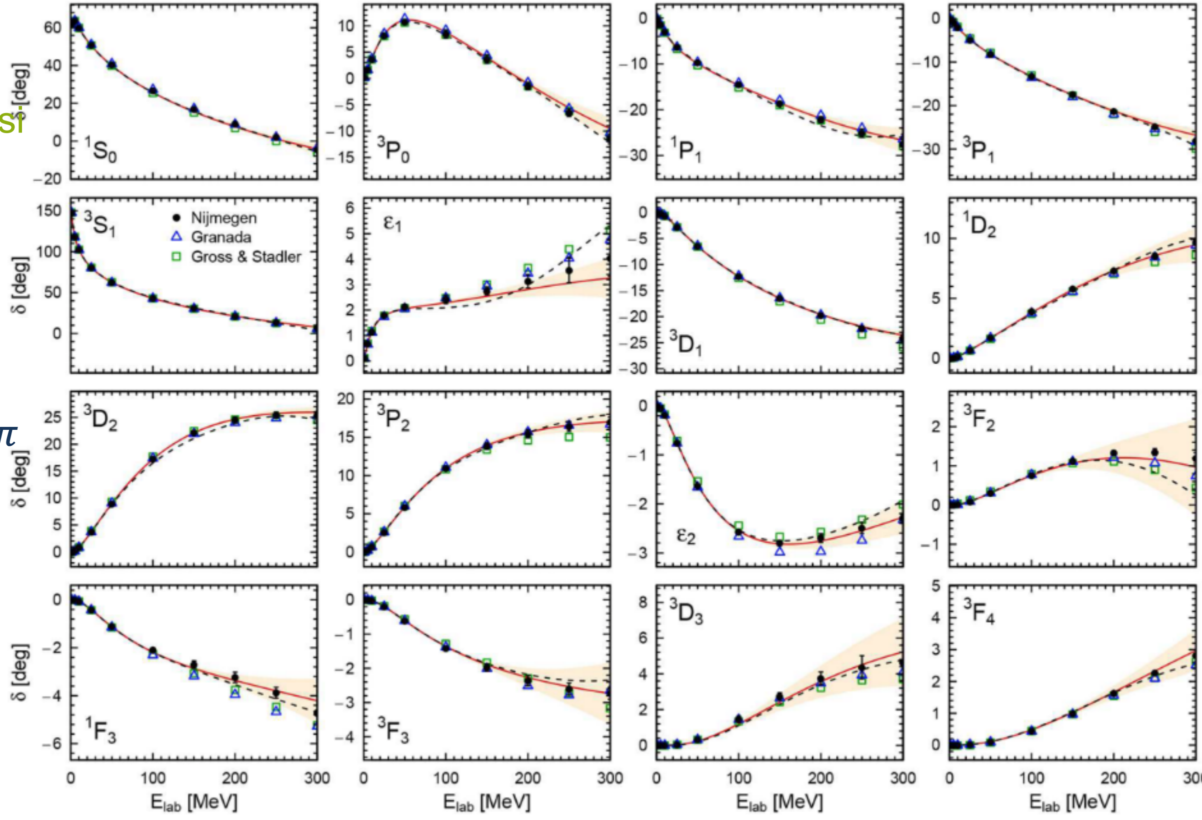
Chiral EFT

$$V(\vec{p}', \vec{p}) = V_{\text{contact}} + V_{1\pi} + V_{2\pi}$$



- Derived in the momentum space, E-independent
- Semilocal momentum-space regularization

Reinert:2017usr



$$V_{1\pi}(\vec{p}', \vec{p}) = -\frac{g_A^2}{4F_\pi^2}\left(\frac{\vec{\sigma}_1 \cdot \vec{q}\vec{\sigma}_2 \cdot \vec{q}}{q^2 + m_\pi^2} + C(m_\pi)\vec{\sigma}_1 \cdot \vec{\sigma}_2\right) e^{-\frac{q^2+m_\pi^2}{\Lambda^2}}$$

- Benefit from the known long-range interaction $V_{1\pi}$
- Low energy constants (LECs) for short-range interaction (contact interaction)
  - ▶ fitting lattice QCD data

Our formalism: plane wave basis expansion

- $|\boldsymbol{p_n}, \boldsymbol{\eta}\rangle$: $\boldsymbol{p_n}$ discrete momentum, $\boldsymbol{\eta}$: polarization vector for $S = 1$

$$\hat{D}(g)|\boldsymbol{p},\boldsymbol{\eta}\rangle = |g\boldsymbol{p}, g\boldsymbol{\eta}\rangle, \hat{P}|\boldsymbol{p},\boldsymbol{\eta}\rangle = |-\boldsymbol{p},\boldsymbol{\eta}\rangle\,, \langle \boldsymbol{p_{n'}}, \boldsymbol{\eta}'^{\dagger}|\hat{D}(g)|\boldsymbol{p_n},\boldsymbol{\eta}\rangle = \delta_{\boldsymbol{n'n}}\left(\boldsymbol{\eta}'^{\dagger}\cdot g\boldsymbol{\eta}\right)$$

- $\{|\boldsymbol{p_n}, \boldsymbol{\eta}\rangle\}$ form the representation space of corresponding point group
- For non-relativistic systems, Lippmann-Schwinger equation (LSE)
  - ▶ matrix equation $\mathbb{T} = \mathbb{V} + \mathbb{VGT}$
  - ▶ Finite volume levels ⇒ Eigenvalue problem

$$\det\left(\mathbb{G}^{-1} - \mathbb{V}\right) = 0 \to \det\left(\mathbb{H} - E\mathbb{I}\right) = 0,$$

  - ▶ Reduce the $\mathbb{H}$ according to irreducible representations (irreps) of the point group

$$\mathbb{H} \Rightarrow \mathrm{diag}\{\mathbb{H}_{\Gamma_i}, \mathbb{H}_{\Gamma_j}, \ldots\} \Rightarrow \quad \mathbb{H}_{\Gamma}\boldsymbol{v} = E_{\Gamma}\boldsymbol{v}$$

- For moving systems, elongated boxes, particles with arbitrary spin...
- dim of the $\mathbb{H}_{\Gamma}$: cubic function

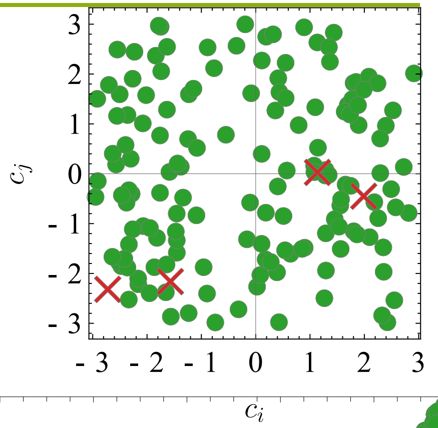
$$\dim \sim \left(\frac{\Lambda_{\mathrm{UV}}}{2\pi/L}\right)^3 \sim \mathcal{O}(1000), \quad \Lambda_{\mathrm{UV}} \approx 4\mathrm{GeV}$$

Similar approaches: Momentum lattice (Doring:2011ip), Hamiltonian EFT(Wu:2014vma, Liu:2015ktc)…

Towards a practical approach: eigenvector continuation

- Plane wave basis+Eigenvector continuation
  - ► Eigenvector continuation (EC) with subspace learning

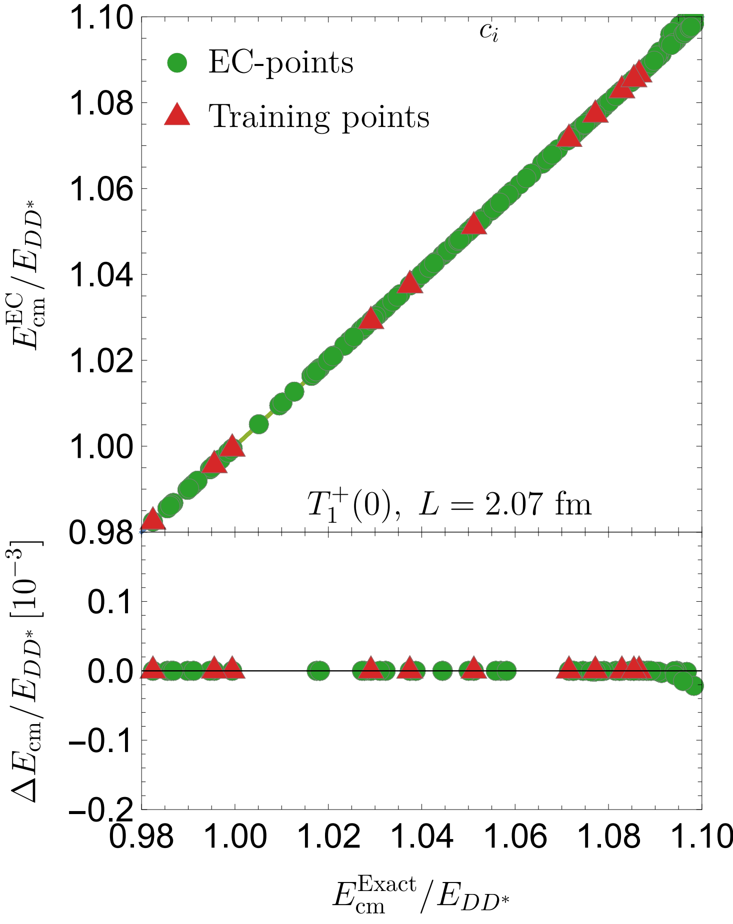
Frame:2017fah,Demol:2019yjt, Furnstahl:2020abp,Yapa:2022nnv



- To fit or quantify uncertainty: solve eigenvalue problem with different $\{c_i\}$ repeatedly
- EC basis: eigenvectors from a selection of parameter sets $\{c_i\}_1$, $\{c_i\}_2$,... (training point)
- Naturalness of LEC in EFT (~ 1) makes the EC more reliable
- dim is linear function

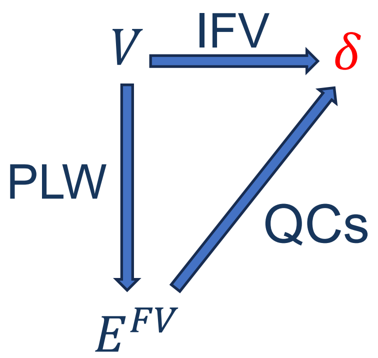
$$\dim^{EC} = \frac{p_{max}}{2\pi/L} \sim \mathcal{O}(10), \quad p_{max} \approx 0.6 \text{ GeV}$$

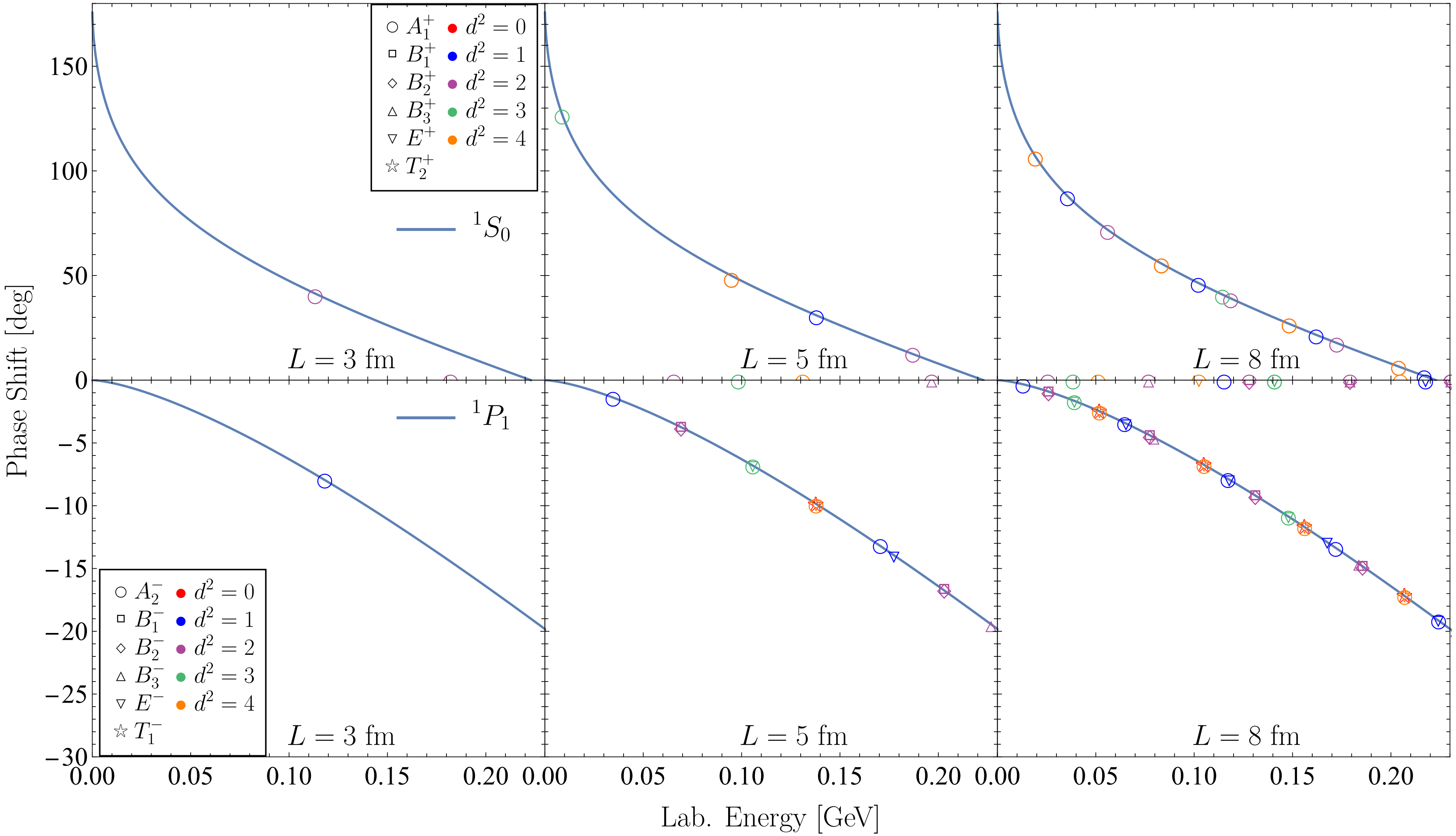
- The subspace learning is the one-time cost
- Make the calculation fast and accurate



Benchmark: contact interaction

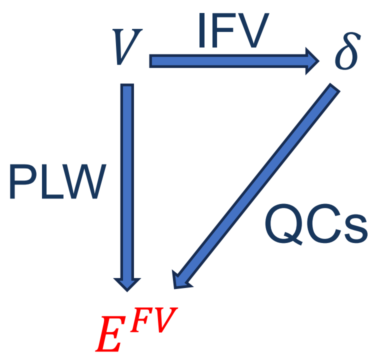
- Contact interaction: $V(\boldsymbol{p},\boldsymbol{p}') = C_S + C_1\boldsymbol{q}^2 + C_2\boldsymbol{k}^2$
- Only contribute to S-wave and P-wave



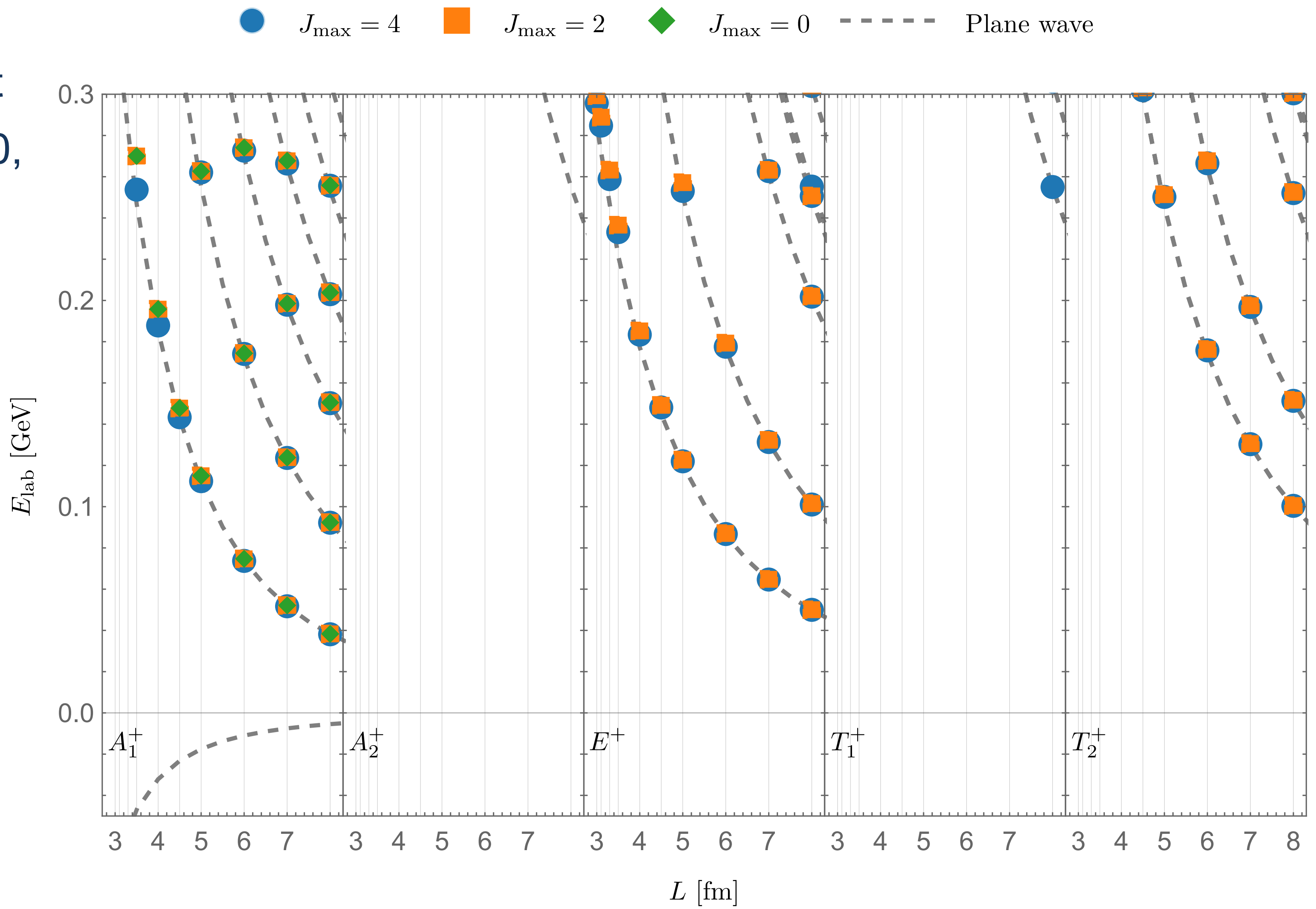


Benchmark: chiral EFT

- ChEFT nuclear force: NNLO
- S=0, $d = (0,0,0)$, even parity
- QCs with partial mixing effect
- L={ 3.0,3.1,3.3,3.5,4.0,4.5,5.0, 6.0,7.0,8.0} fm

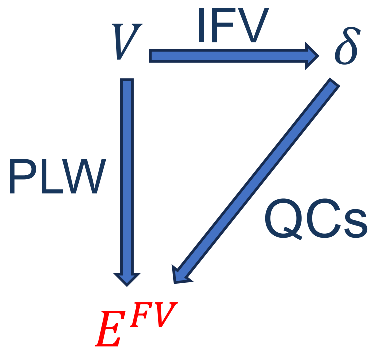


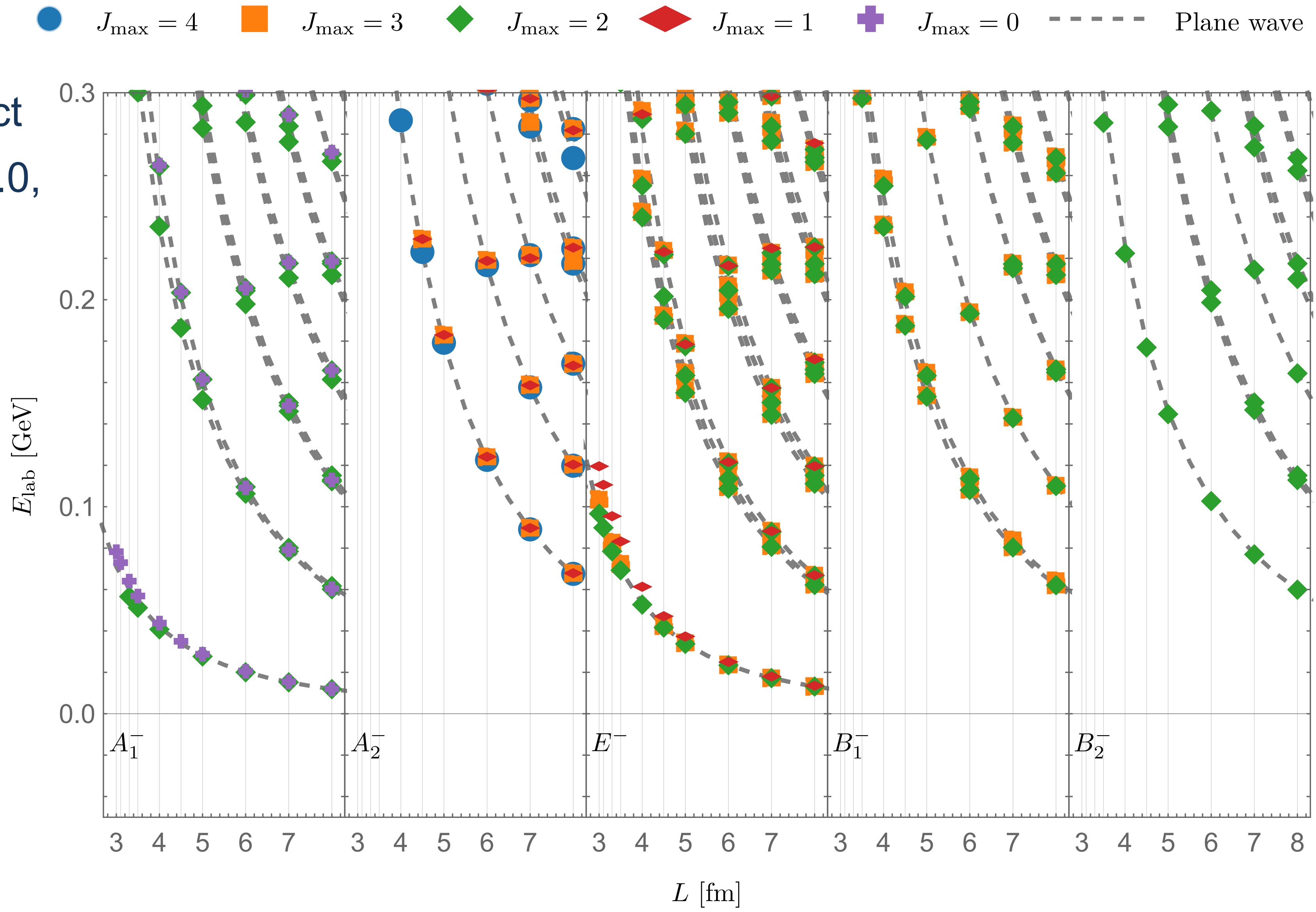
- The discrepancy
  - ▶ Small box
  - ▶ Small J_{max} truncation



Benchmark: chiral EFT

- ChEFT nuclear force: NNLO
- S=1, d=(0,0,1), odd parity
- QCs with partial mixing effect
- L={ 3.0,3.1,3.3,3.5,4.0,4.5,5.0, 6.0,7.0,8.0} fm



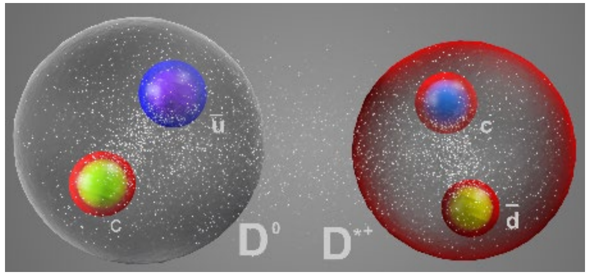


- The discrepancy
  - ▶ Small box
  - ▶ Small $J_{\max}$ truncation

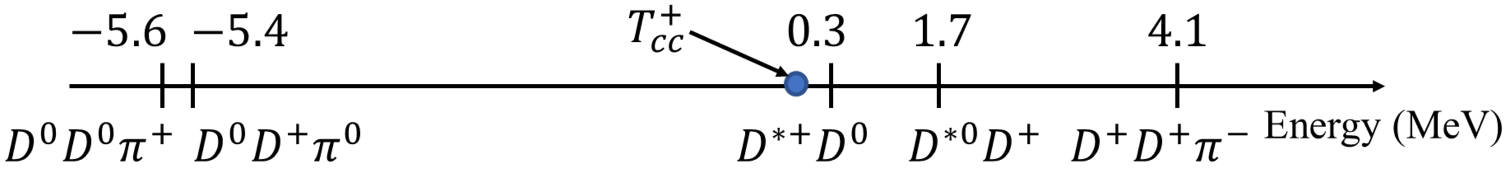
$T_{cc}(3875)^+$ state

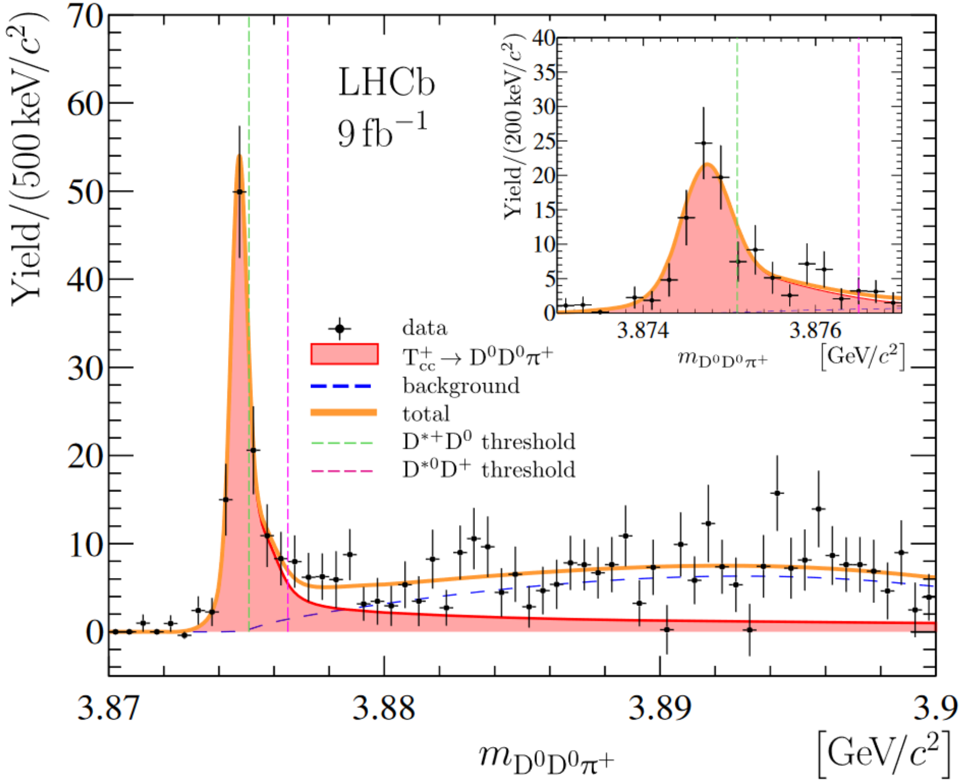
- $T_{cc}(3875)^+$ was observed in 3-body final states: $D^0D^0\pi^+$
- Very close to D^0D^{*+} thresholds: $\delta m_U \approx -360$keV, $\Gamma \approx 48$ keV
- Exotic hadrons: minimal quark content: $cc\bar{u}\bar{d}$
- Good candidates of D^0D^{*+} molecule

LHCb Collaboration



- 3-body dynamics could be important





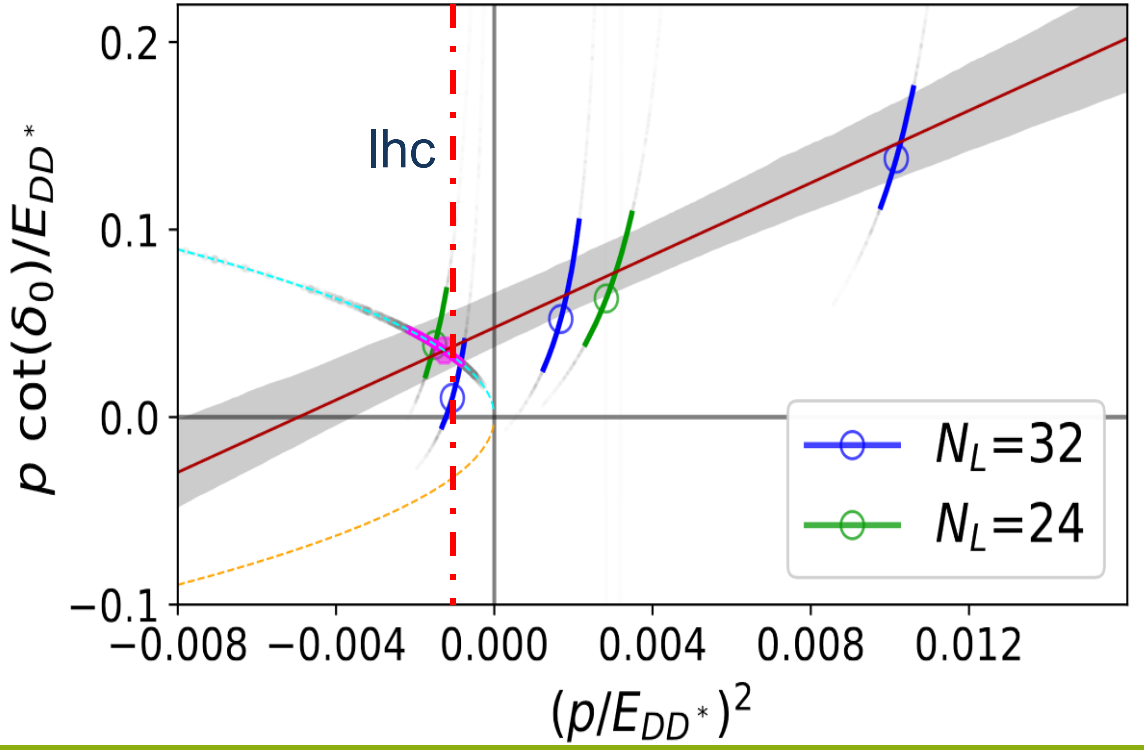
LHCb:2021vvq, LHCb:2021auc, Du:2021zzh, Meng:2021jnw…

Padmanath:2022cvl

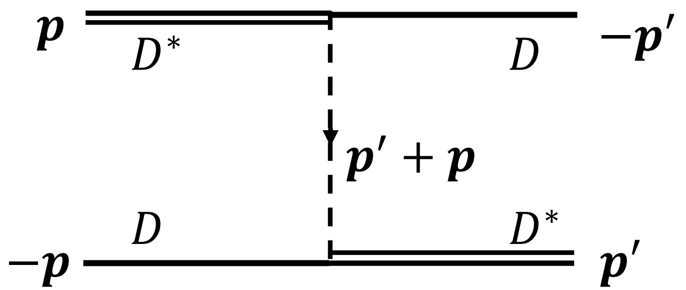
- LQCD: $m_\pi \approx$ 280 MeV, $m_D \approx 1927$ MeV, $m_{D^*} \approx 2049$ MeV, $L \approx 2.07, 2.76$ fm, $a \approx 0.086$fm
- Some quick estimations
  - $m_{\text{eff}}^2 = m_\pi^2 - (m_{D^*} - m_D)^2 > 0$, $m_{\text{eff}} \approx 252$ MeV
  - $m_{\text{eff}} L = 2.6, 3.5$
  - $p_{\text{lhc}}^2 \approx -\left(\frac{m_{\text{eff}}}{2}\right)^2 = -(126\ \text{MeV})^2$
  - $p_{\text{rhc3}}^2 \approx 2\mu_{DD^*}(2m_D + m_\pi - m_D - m_{D^*}) \approx (560\ \text{MeV})^2$

① e^{-mL} effect can be neglected
② Considering the PW mixing effect
③ E^{FV} well above lhc

④ERE works in IFV Du:2023hlu



rhc3

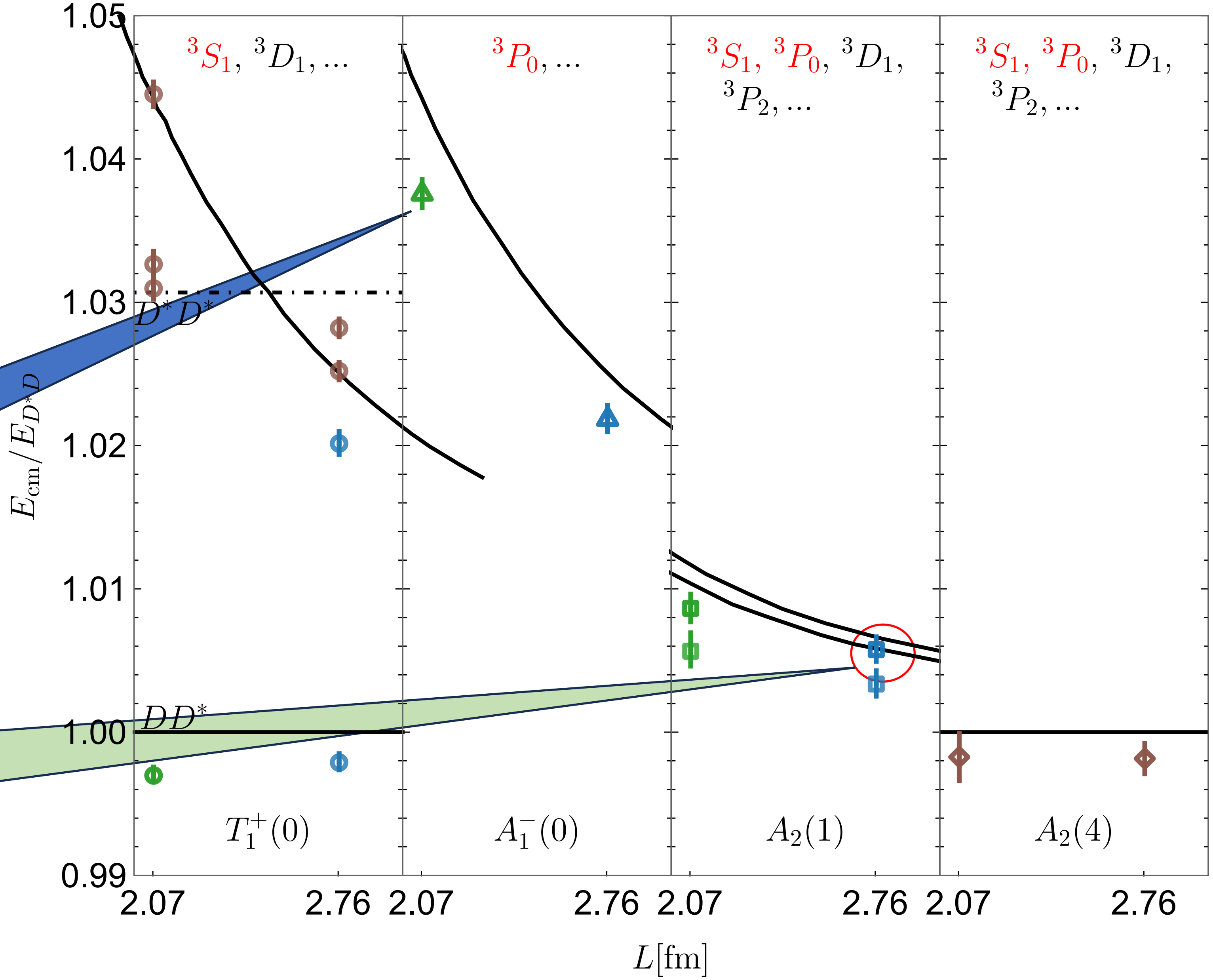


$$\frac{1}{k^2 - m_\pi^2 + i\epsilon} = \frac{-1}{\boldsymbol{k}^2 + m_\pi^2 - k_0^2 - i\epsilon} = \frac{-1}{\boldsymbol{k}^2 + m_{\text{eff}}^2 - i\epsilon}$$

Other lattice results: Cheung:2017tnt, Junnarkar:2018twb, Chen:2022vpo, Lyu:2023xro

Lattice FV energy levels

- Green and blue points as input
- 9 inputs in total



Highest input: $p \approx 0.56$ GeV
- relativistic effect
- large cutoff $\Lambda \geq 0.7$

Overlap with noninteracting energy levels: $\cot 0 = \infty$;

Singular in $\mathrm{pcot}\,\delta$ figures

Relativistic formalism

- The energy levels of $A_1^-(0)$ is high, relativistic formalism

$$T(\boldsymbol{p},\boldsymbol{p}') = V(\boldsymbol{p},\boldsymbol{p}') + \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} V(\boldsymbol{p},\boldsymbol{q}) \frac{1}{2w_1 w_2} \frac{(w_1+w_2)}{P_0^2-(w_1+w_2)^2+i\epsilon} T(\boldsymbol{q},\boldsymbol{p}')$$

$$w_i = \sqrt{m_i^2+\boldsymbol{q}^2}$$

$$G = i\int \frac{d^4q}{(2\pi)^4} \frac{1}{(P-q)^2-m_1^2+i\epsilon} \frac{1}{q^2-m_2^2+i\epsilon}$$

- Replace integral into summation to get $\mathbb{T} = \mathbb{V} + \mathcal{J}\mathbb{V}.\mathbb{G}.\mathbb{T}$

$$\int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \to \mathcal{J}\int \frac{d^3\boldsymbol{q}_{box}}{(2\pi)^3} \to \mathcal{J}\sum_{\boldsymbol{n}} \frac{1}{L^3}$$

$\mathcal{J}$: is the Jacobian determinant of the Lorentz boost

Li:2021mob

- Get the poles

$$\det(\mathbb{H}-\lambda\mathbb{I}) = 0 \to \mathbb{H}\boldsymbol{v} = \lambda\boldsymbol{v},$$

Interaction: ChEFT

- Contact terms to NLO

$\mathcal{O}_1$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_4$	$\mathcal{O}_5$	$\mathcal{O}_6$
$\boldsymbol{\epsilon}'^{\dagger}\cdot\boldsymbol{\epsilon}$	$\boldsymbol{q}^2\boldsymbol{\epsilon}'^{\dagger}\cdot\boldsymbol{\epsilon}$	$4\boldsymbol{k}^2\boldsymbol{\epsilon}'^{\dagger}\cdot\boldsymbol{\epsilon}$	$4\boldsymbol{k}\cdot\boldsymbol{\epsilon}'^{\dagger}\boldsymbol{k}\cdot\boldsymbol{\epsilon}$	$\boldsymbol{q}\cdot\boldsymbol{\epsilon}'^{\dagger}\boldsymbol{q}\cdot\boldsymbol{\epsilon}$	$(\boldsymbol{\epsilon}'^{\dagger}\times\boldsymbol{\epsilon})\cdot(\boldsymbol{q}\times 2\boldsymbol{k})$

$$\boldsymbol{q} = \boldsymbol{p}' - \boldsymbol{p},$$
$$\boldsymbol{k} = \frac{\boldsymbol{p}' + \boldsymbol{p}}{2}$$

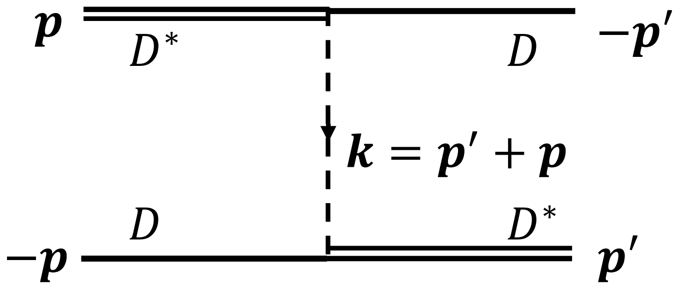
$$\mathcal{O}^1_{^3S_1} = \mathcal{O}_1$$
$$\mathcal{O}^2_{^3S_1} = \mathcal{O}_2 + \mathcal{O}_3$$
$$\mathcal{O}_{^3S_1-^3D_1} = -\mathcal{O}_2 - \mathcal{O}_3 + 3\mathcal{O}_4 + 3\mathcal{O}_5$$
$$\mathcal{O}_{^3P_0} = -(\mathcal{O}_4 - \mathcal{O}_5) + \mathcal{O}_6$$
$$\mathcal{O}_{^3P_1} = -\tfrac{3}{2}(\mathcal{O}_2 - \mathcal{O}_3) + \tfrac{3}{2}(\mathcal{O}_4 - \mathcal{O}_5) + \tfrac{3}{2}\mathcal{O}_6$$
$$\mathcal{O}_{^3P_2} = -\tfrac{3}{2}(\mathcal{O}_2 - \mathcal{O}_3) - \tfrac{1}{2}(\mathcal{O}_4 - \mathcal{O}_5) - \tfrac{5}{2}\mathcal{O}_6$$

- In present calculation: LO and NLO $^3S_1$ contact terms, NLO $^3P_0$
  - $^3S_1$- $^3D_1$ transition term and $^3P_2$ are included to estimate the systemic uncertainties
- Separable regulator: $e^{-\frac{p^n+p'^n}{\Lambda^n}}$, $n = 2,4,6$

One-pion-exchange interaction

- Pion propagator: static approximation

$$D \approx -\boldsymbol{k}^2 - \left[m_\pi^2 - (M_{D^*} - M_D)^2\right] + i\epsilon$$



- Semilocal momentum-space regularization

$$\mathcal{V}(k) = -\frac{g^2}{4F_\pi}\left[\frac{\boldsymbol{k}\cdot\boldsymbol{\epsilon}'^*\boldsymbol{k}\cdot\boldsymbol{\epsilon}}{\boldsymbol{k}^2+u^2} + C_{sub}\boldsymbol{\epsilon}'^*\cdot\boldsymbol{\epsilon}\right] e^{-\frac{k^2+u^2}{\Lambda^2}}$$

$$C_{sub} = -\frac{\Lambda(\Lambda^2 - 2u^2) + 2\sqrt{\pi}u^3 e^{\frac{u^2}{\Lambda^2}}\operatorname{erfc}(\frac{u}{\Lambda})}{3\Lambda^3}$$

  - ▶ The regulator will not change the long-range behavior
  - ▶ The short-range part of OPE is subtracted: $V_{\epsilon\cdot\epsilon}(r=0) = 0$

Reinert:2017usi

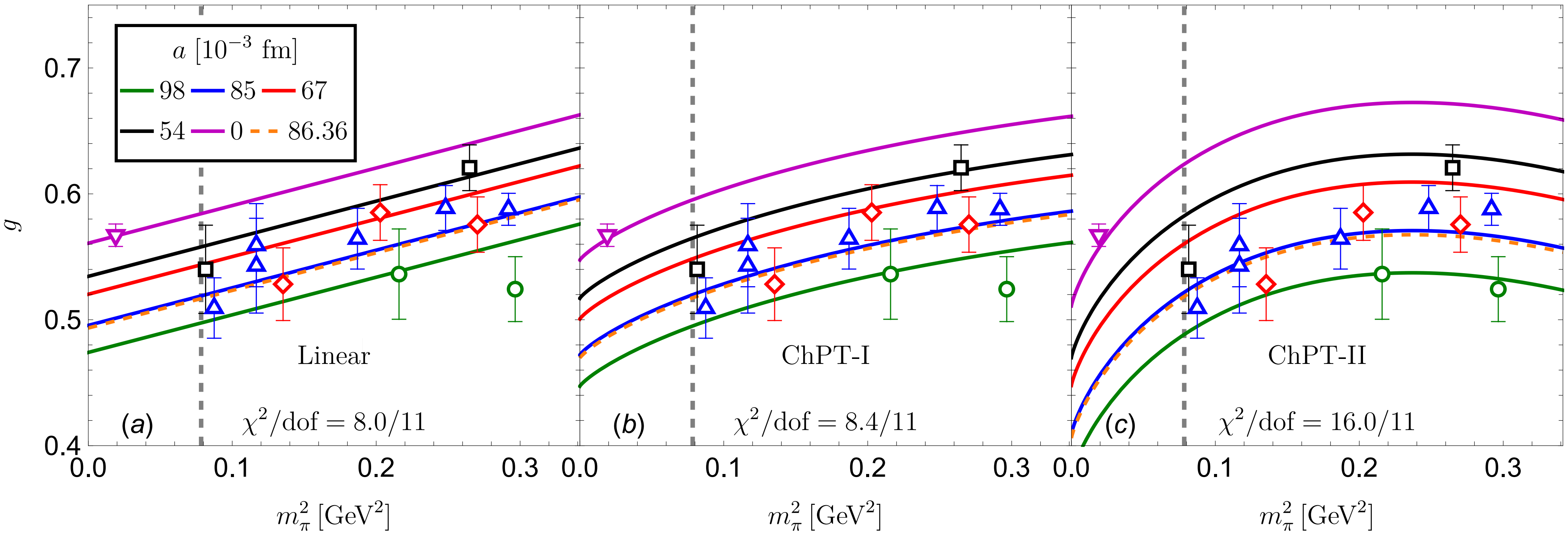
F_π and g

- F_π and $g_{D^*D\pi}$ at $m_\pi = 280$ MeV are determined by lattice QCD data, physical values by either linear extrapolation or chiral extrapolation
- $F_{ph} = 92.1$ MeV, $F_0 = 85$ MeV, chiral extrapolation, $\xi = m/m^{ph}$

$$f_\pi(\xi) = f_\pi^{\rm ph}\left[1 + \left(1 - \frac{f_0}{f_\pi^{\rm ph}}\right)(\xi^2 - 1) - \frac{(m_\pi^{\rm ph})^2}{8\pi^2 f_0^2}\xi^2 \log\xi\right]$$

Du:2023hlu,Becirevic:2012pf

- Three extrapolations give the consistent results
  - ▶ The g is slightly smaller than the value in Ref. [Du:2023hlu]
  - ▶ $g = 0.517 \pm 0.015$ for $a = 0.086$ fm



$\Lambda = 0.9$ GeV, only contact terms

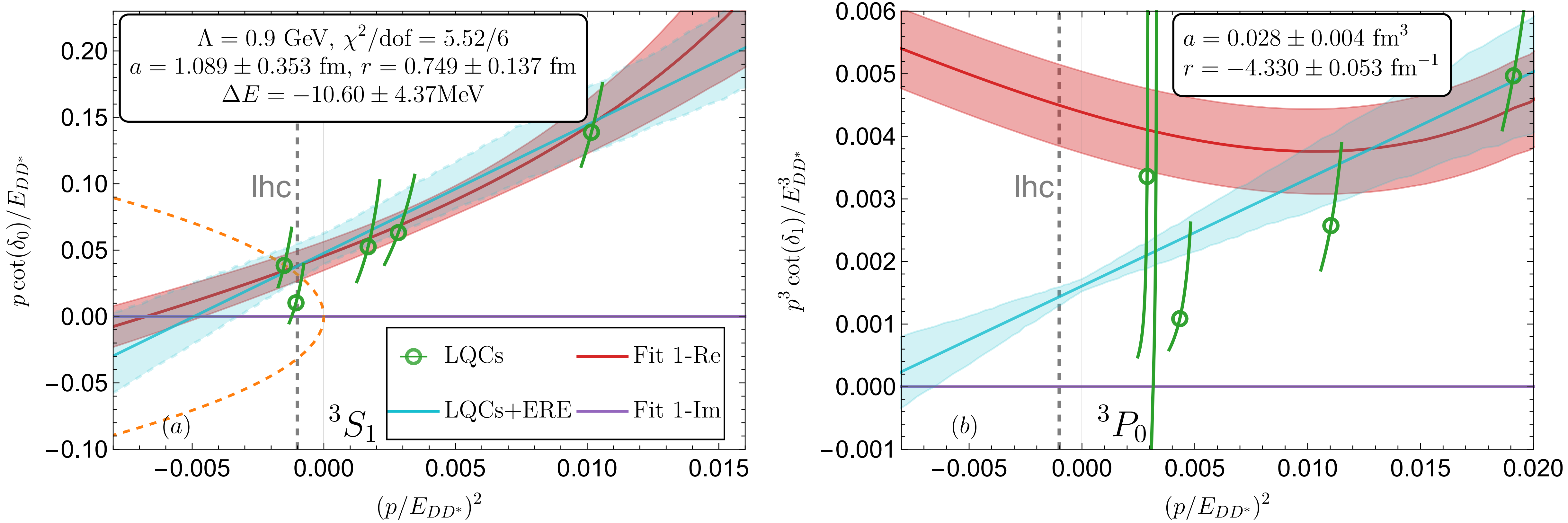
- Results using Lüscher's QCs Padmanath:2022cvl

$\chi^2/\text{dof}=3.7/5,\ E_{\text{pole}}^{^3S_1} = -9.9^{+3.6}_{-7.2}$ MeV

$a_{\,^3S_1} = 1.04(29)\text{fm}, \quad r_{\,^3S_1} = 0.96^{+0.18}_{-0.20}\text{fm}$

$a_{\,^3P_0} = 0.076^{+0.008}_{-0.009}\text{fm}^3, \ r_{\,^3P_0} = 6.9(2.1)\text{fm}^{-1}$

- Our results



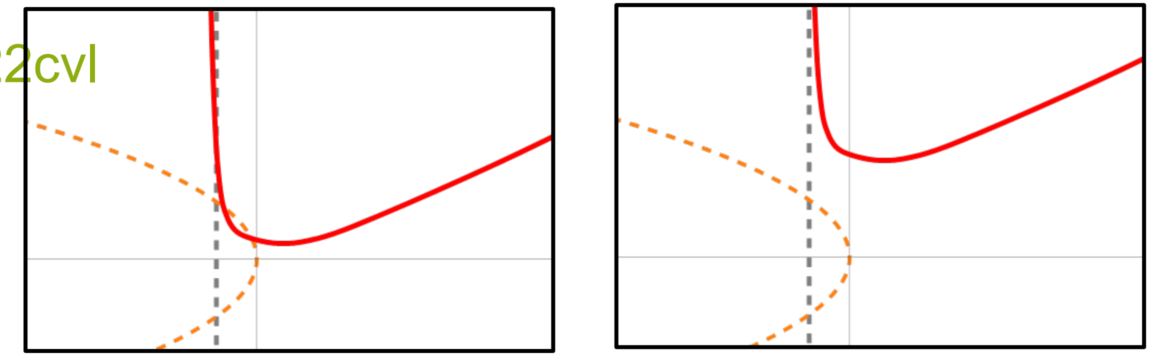
$\Lambda = 0.9$ GeV, contact terms+OPE

- Results using Lüscher's QCs Padmanath:2022cvl

$\chi^2/\text{dof}=3.7/5,\ E_{\text{pole}}^{^3S_1} = -9.9^{+3.6}_{-7.2}$ MeV

$a_{^3S_1} = 1.04(29)\text{fm}, \quad r_{^3S_1} = 0.96^{+0.18}_{-0.20}\text{fm}$

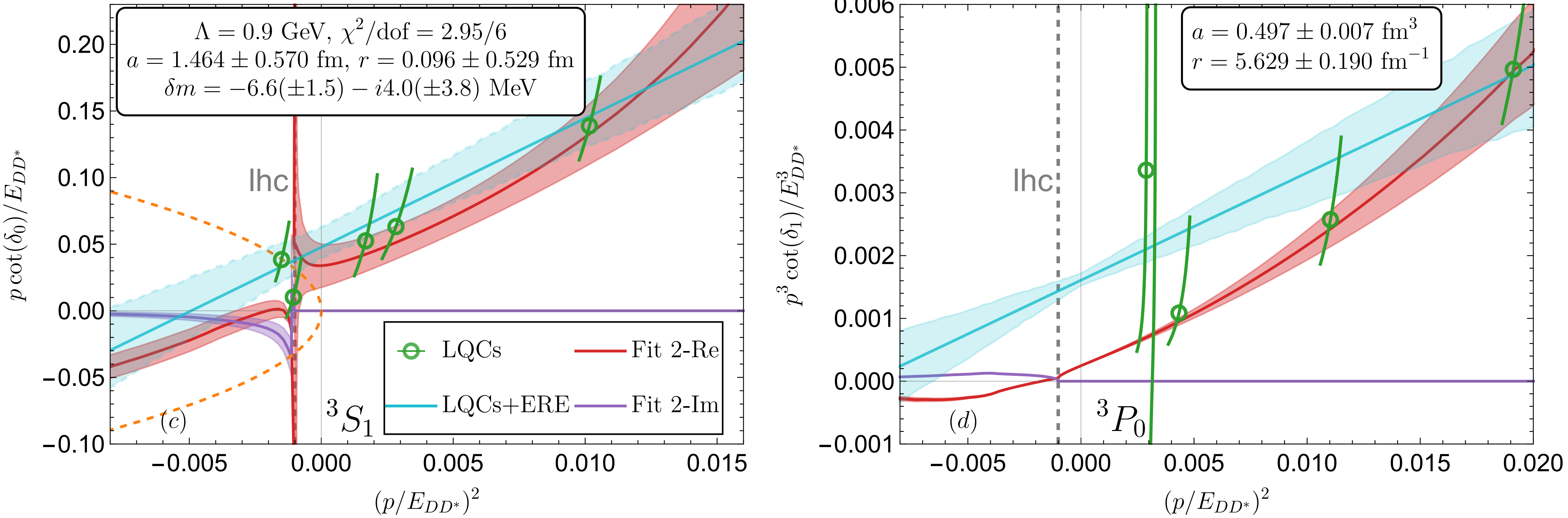
$a_{^3P_0} = 0.076^{+0.008}_{-0.009}\text{fm}^3, \ r_{^3P_0} = 6.9(2.1)\text{fm}^{-1}$



two virtual states   Resonance

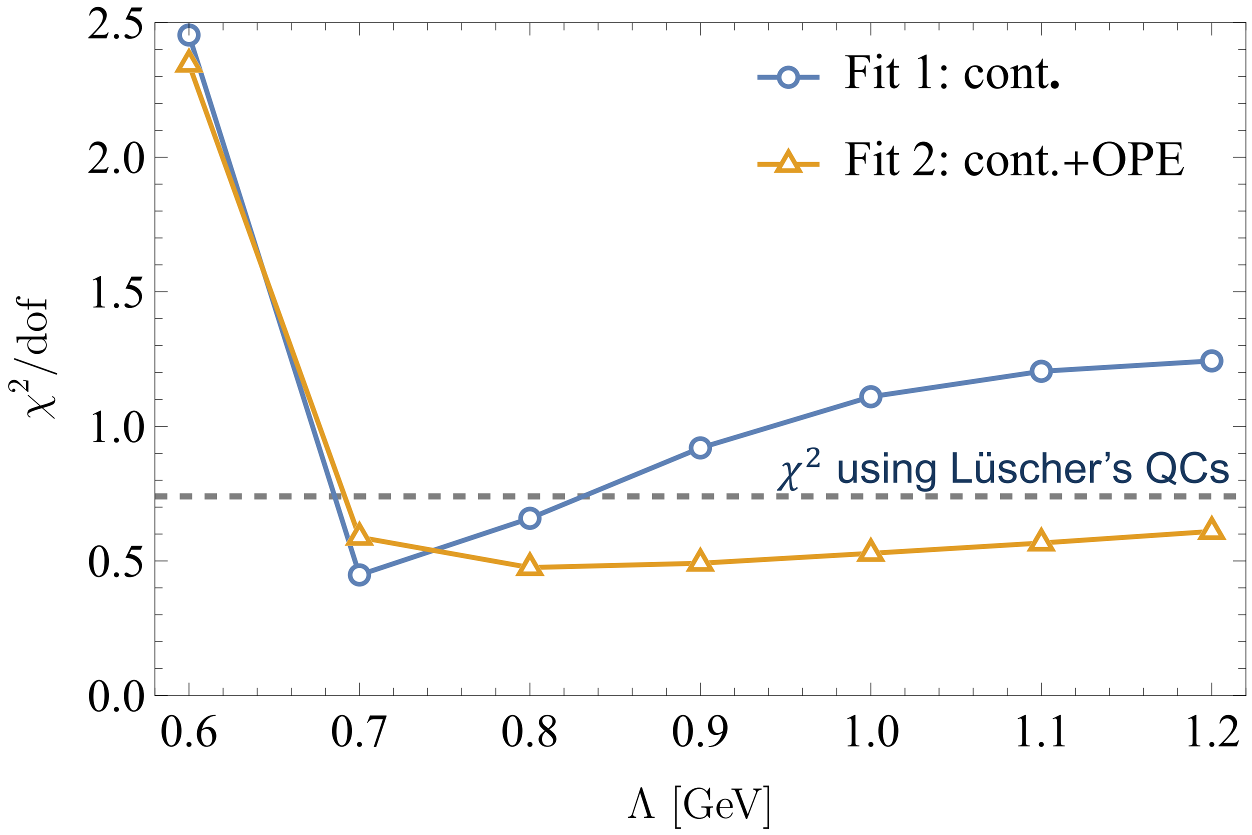
- Our results

Resonance with 85% probability within the 1σ uncertainty

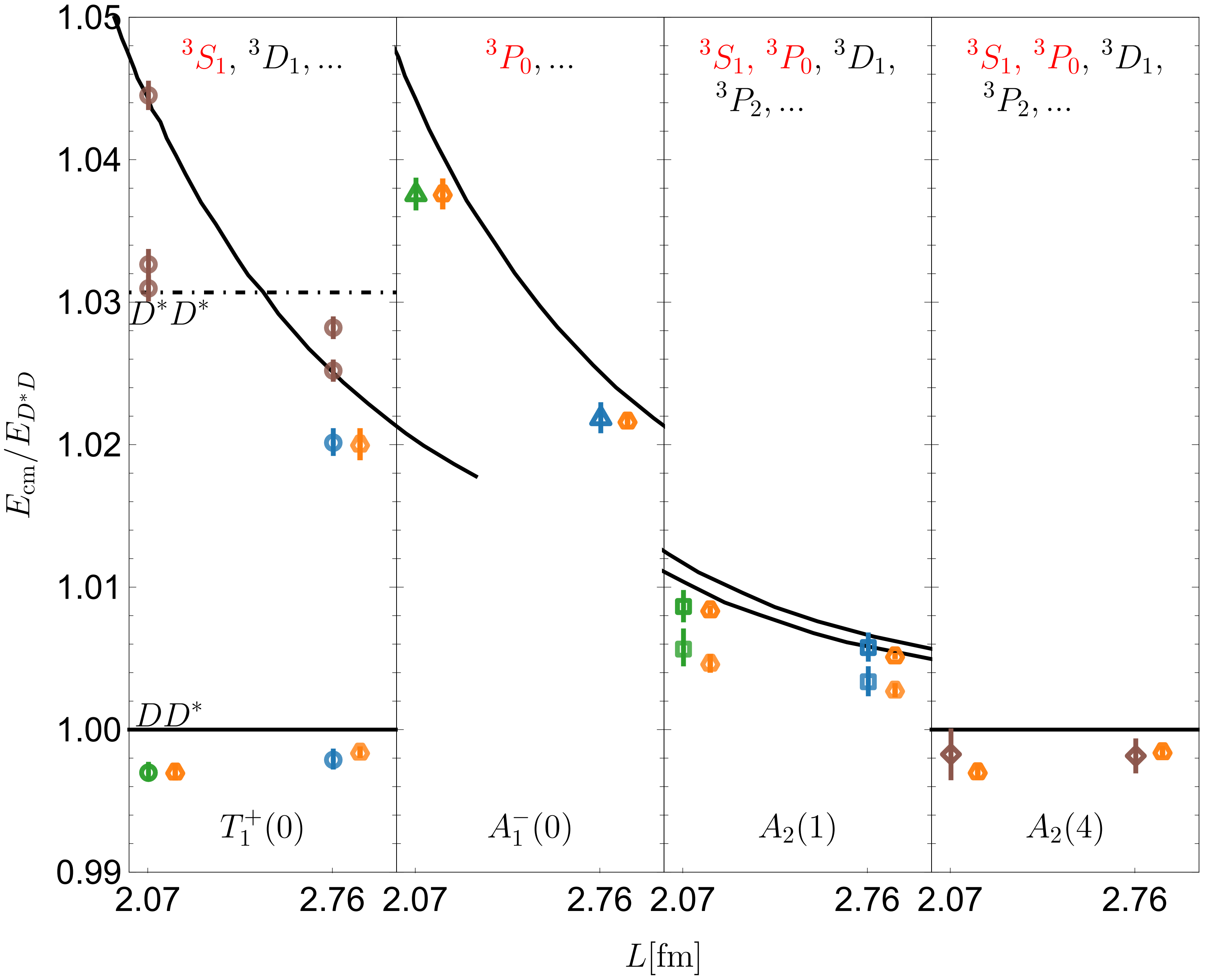


Cutoff dependence of χ^2

- 3 LECs: LO and NLO $^3S_1$ contact terms, NLO $^3P_0$
- In $V_{\rm ctc}$ fit, the P-wave dominate states control Λ-dependence of the χ^2
  - ▶ The shape of the of $k^3\cot\delta_1$ is determined by regulator and cutoff
  - ▶ Sensitive to Λ
- The $V_{\rm ctc}$+ $V_{1\pi}$ fit is stable with Λ
- The $V_{\rm ctc}$+ $V_{1\pi}$ fit is even better than QCs

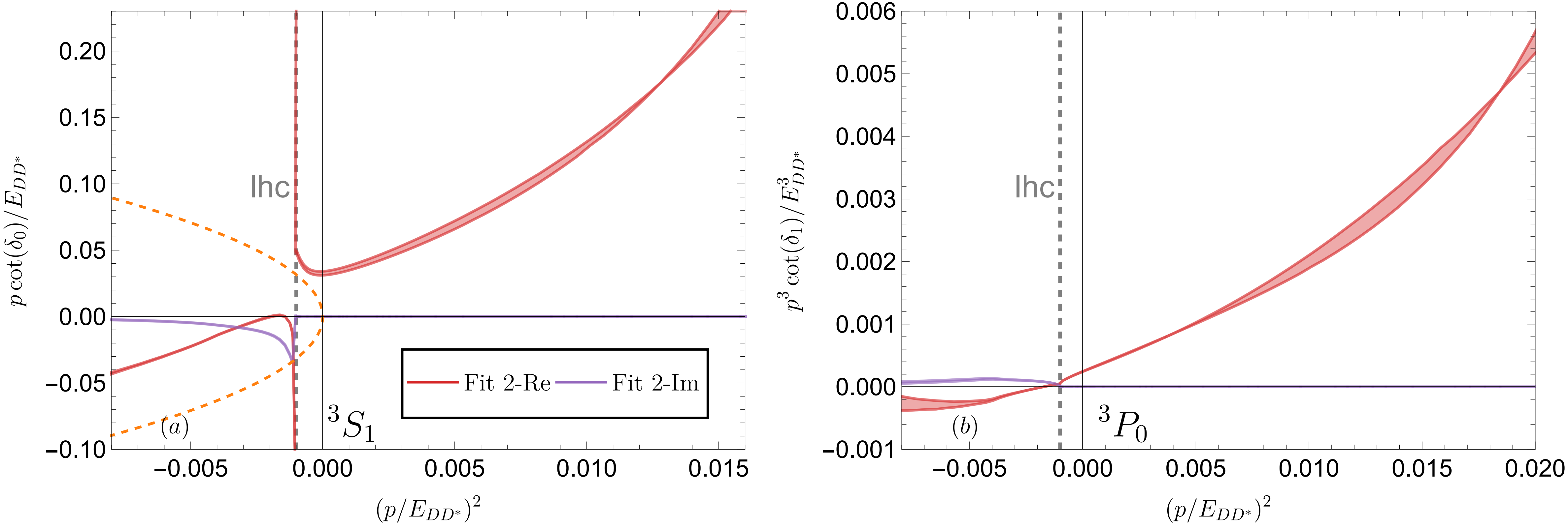


Energy levels from the best fit

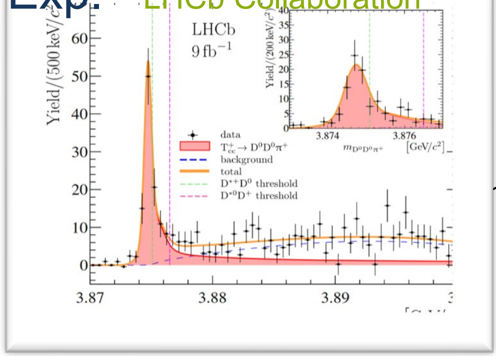


Cutoff dependence of phase shift

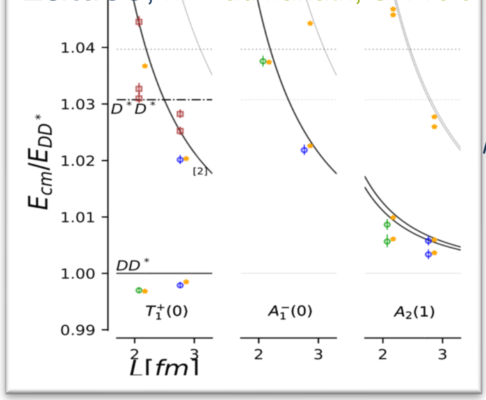
$\Lambda = 0.7 - 1.2$ GeV



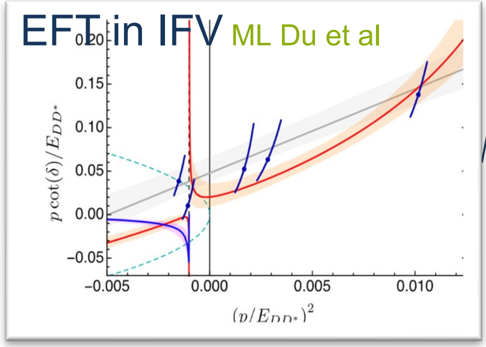
Exp. LHCb Collaboration



Lattice, M. Padmanath, S. Prelovsek



EFT in IFV ML Du et al



2108.02709

2109.01038

2202.10110

2303.09441

Lattice2023

2311.18793

2312.01930

Lattice2021

Summary

- A proof-of-principle of an alternative method of Lüscher's formu
- LSE or BSE in plane wave expansion+projection operator technique reduction to irreps
  - ⇒ Including partial wave mixing effect naturally, avoid complications of PW expansion
  - ⇒ Rest and moving two-particle systems, spinless, equal mass
- Non-relativistic example: spin-triplet NN
  - ⇒ S-wave dominant states: LF works well for $L \gtrsim 5$ fm
  - ⇒ P-wave dominant states: OPE→large PW mixing effect regardless the box size
  - ⇒ EFT-based approach in the plane wave basis: $V_{1\pi} + V_{1h} \xrightarrow[\text{data}]{\text{fit}} V_{1\pi} + V^{(0)}_{\text{cont}} + V^{(2)}_{\text{cont}} \cdots$
  - ⇒ Advantages: 1) insensitive to PW mixing artifact; 2) small box (long-range interaction)

g the plane wave basis

EFT in FV

Finite-volume scattering on the left-hand cut *Curia II, WH2SW*	*Andre Baiao Raposo*	13:30 - 13:50
Breakdown of Lüscher Formalism near Left Hand Cuts *Curia II, WH2SW*	*Md Habib E Islam*	13:50 - 14:10
Resolving the left-hand-cut problem in lattice studies of the doubly-charmed tetraquark *Curia II, WH2SW*	*Steve Sharpe*	14:10 - 14:30

Finite-volume scattering on the left-hand cut

André Baião Raposo, Maxwell T. Hansen

Nov 30, 2023

59 pages

e-Print: 2311.18793 [hep-lat]

Modified Lüscher QCs

framework and toy models

Solving the left-hand cut problem in lattice QCD: $T_{cc}(3875)^+$ from finite volume energy levels

Lu Meng, Vadim Baru, Evgeny Epelbaum, Arseniy A. Filin, Ashot M. Gasparyan

Dec 4, 2023

13 pages

e-Print: 2312.01930 [hep-lat]

EFT in FV

Application in Tcc

Two approaches

- Hansen's approach   Raposo:2023oru
  - ▶ FV: lattice data fix $\overline{\mathcal{K}}^{os}$

$$\det_{\boldsymbol{k}^\star \ell m}\left[S(P_j,L)^{-1}+\xi^\dagger\,\overline{\mathcal{K}}^{\text{os}}(P_j)\,\xi+2g^2\mathcal{T}(P_j)\right]=0$$

  - ▶ IFV: solve a integral equation

$$\mathcal{M}^{\text{aux}}(P,p,p')=\mathcal{K}^{\mathcal{T}}(P,p,p')-\frac{1}{2}\int\frac{d^3\boldsymbol{k}^\star}{(2\pi)^3}\frac{\mathcal{M}^{\text{aux}}(P,p,k)\,H(\boldsymbol{k}^\star)\,\mathcal{K}^{\mathcal{T}}(P,k,p')}{4\omega_N(\boldsymbol{k}^\star)\left[(k^\star_{\text{os}})^2-(\boldsymbol{k}^\star)^2+i\epsilon\right]},$$

$$\mathcal{K}^{\mathcal{T}}(P,p,p')=\overline{\mathcal{K}}^{\text{os}}(P,p,p')+2g^2\mathcal{T}(P,p,p'),$$

$$S_{\boldsymbol{k}^\star\ell m,\boldsymbol{k}'^\star\ell'm'}(P,L)=\frac{1}{2L^3}\frac{4\pi\,Y_{\ell m}(\hat{\boldsymbol{k}}^\star)\,Y^*_{\ell'm'}(\hat{\boldsymbol{k}}^\star)\,\delta_{\boldsymbol{k}^\star\boldsymbol{k}'^\star}\,|\boldsymbol{k}^\star|^{\ell+\ell'}\,e^{-\alpha[(\boldsymbol{k}^\star)^2-(k^\star_{\text{os}})^2]}}{4\omega_N(\boldsymbol{k})\left[(k^\star_{\text{os}})^2-(\boldsymbol{k}^\star)^2\right]},$$

$$\mathcal{T}_{\boldsymbol{k}^\star\ell m,\boldsymbol{k}'^\star\ell'm'}(P)=-\frac{1}{4\pi|\boldsymbol{k}^*|^\ell|\boldsymbol{k}'^*|^{\ell'}}\int d\Omega_{\hat{\boldsymbol{k}}^*}d\Omega_{\hat{\boldsymbol{k}}'^*}Y_{\ell m}(\hat{\boldsymbol{k}}^*)Y^*_{\ell'm'}(\hat{\boldsymbol{k}'^*})\times\frac{1}{(p'-p)^2-M_\pi^2}$$

$$\omega_N(\boldsymbol{p})^2=m_N^2+\boldsymbol{p}^2,\quad p=(\omega_N(\boldsymbol{k}^*),\ \boldsymbol{k}^*),\quad p'=(\omega_N(\boldsymbol{k}^{*\prime}),\ \boldsymbol{k}^{*\prime})$$

$\xi=1$

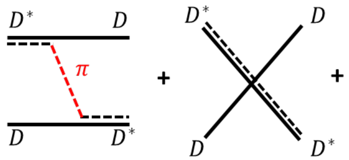
Model-independent?
You have to choose a parameterization of $\overline{\mathcal{K}}^{os}$: ERE
To some how, the ERE is equivalent to the contact EFT

- Our approach
  - ▶ FV: lattice data fix contact terms

$$\det[\mathbb{G}^{-1}(E)-\mathbb{V}]=0.$$

  - ▶ IFV: solve a integral equation

$$T(\boldsymbol{p},\boldsymbol{p}',E)=V(\boldsymbol{p},\boldsymbol{p}')+\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}V(\boldsymbol{p},\boldsymbol{q})G(\boldsymbol{q},E)T(\boldsymbol{q},\boldsymbol{p}',E).$$

$V_{EFT}=$  + ...

$$\mathbb{G}(E)=\frac{\mathcal{J}(\boldsymbol{q}_n)}{L^3}G(\boldsymbol{q}_n,E)\delta_{\boldsymbol{n}',\boldsymbol{n}},\qquad\mathbb{V}=V(\boldsymbol{q}_n,\boldsymbol{q}_{n'})$$

$$G(\boldsymbol{q},E)=i\int\frac{dq^0}{2\pi}\frac{1}{(P-q)^2-m_1^2+i\epsilon}\frac{1}{q^2-m_2^2+i\epsilon}$$
$$=\frac{1}{4\omega_1\omega_2}\left(\frac{1}{E-\omega_1-\omega_2}-\frac{1}{E+\omega_1+\omega_2}\right)$$
$$=\frac{1}{2\omega_1\omega_2}\frac{(\omega_1+\omega_2)}{E^2-(\omega_1+\omega_2)^2+i\epsilon},$$

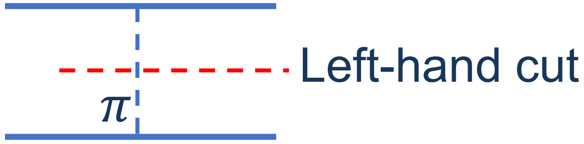
Summary and Outlook

- Validation of Lüscher's formula

  ① e^{-mL} effect can be neglected
  ② Considering the PW mixing effect
  ③ E^{FV} well above lhc
  ④ERE works in IFV

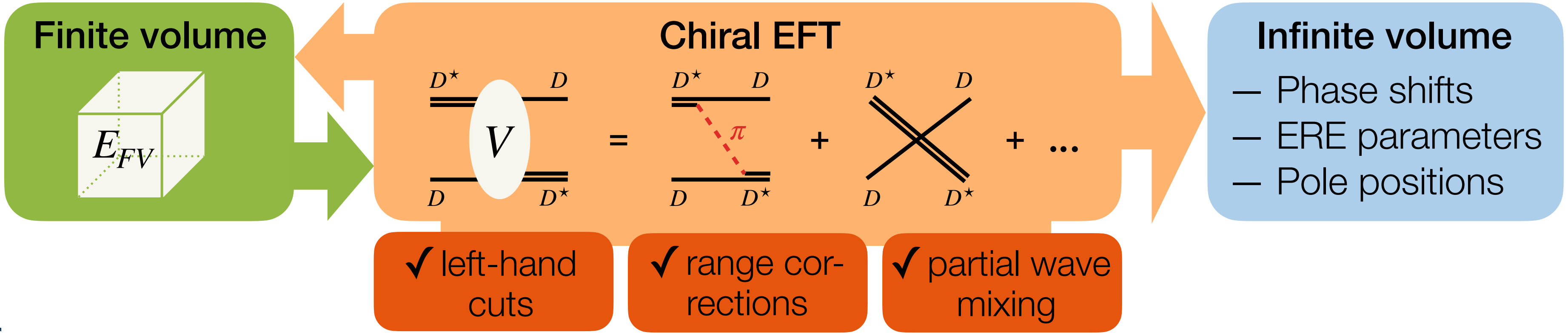
  Du:2023hlu

Invalidate



One-pion-exchange interaction

- Our formalism



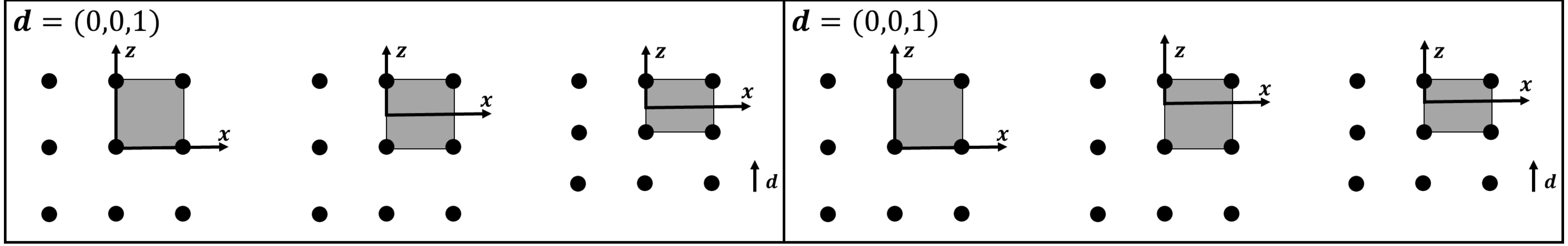
- T_{cc} lattice data
  - ▶ Better fit than QCs
  - ▶ The possible partial wave mixing effect
  - ▶ E^{FV} below the left-hand cut
- Outlook
  - ▶ NN systems
  - ▶ With the more accurate lattice data, we can extract g directly

Thanks for your attentions!

Back up

Moving systems

$m_1 = m_2, \quad A = 1$			$m_1 \neq m_2, \quad A = 1 + \frac{m_1^2 - m_2^2}{E^*}$		
$\boldsymbol{n} \in Z$	$\boldsymbol{n} - \frac{1}{2}\boldsymbol{d}$	$\gamma^{-1}\left(\boldsymbol{n}_\parallel - \frac{\boldsymbol{d}}{2}\right) + \boldsymbol{n}_\perp$	$\boldsymbol{n} \in Z$	$\boldsymbol{n} - \frac{A}{2}\boldsymbol{d}$	$\gamma^{-1}\left(\boldsymbol{n}_\parallel - \frac{A}{2}\boldsymbol{d}\right) + \boldsymbol{n}_\perp$



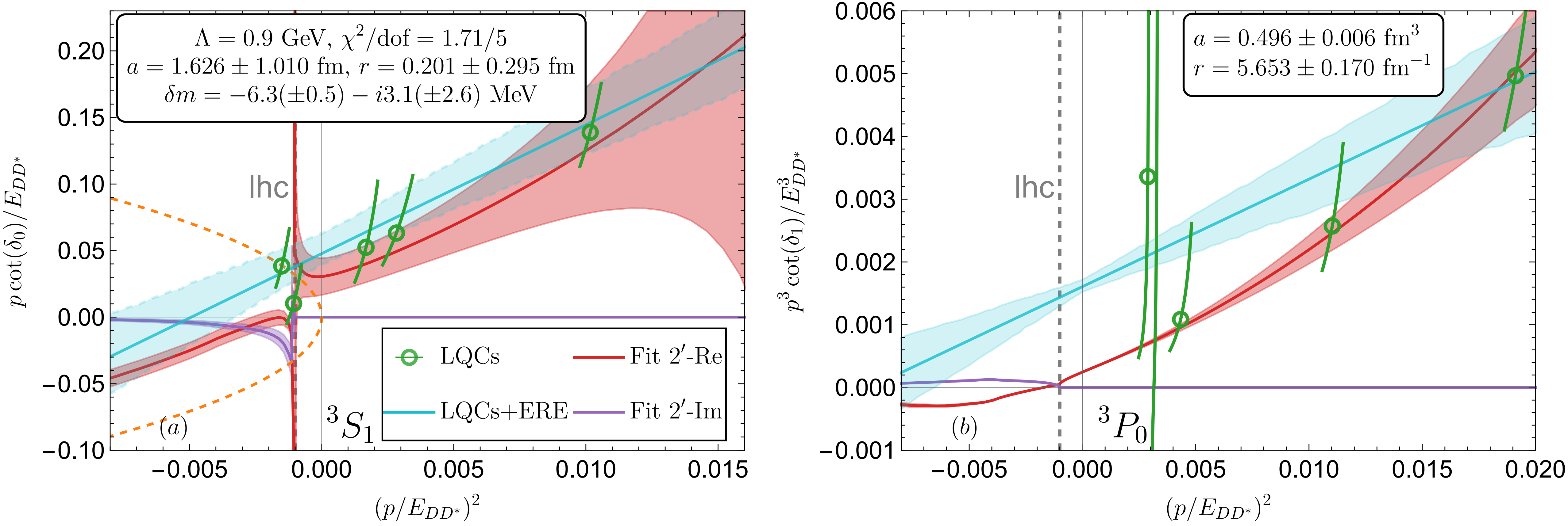
Space inversion invariance is broken

- Moving system in the box $\boldsymbol{P} = \frac{2\pi}{L}\boldsymbol{d} \neq 0$
  - ▶ For LQCD, changing box size is expensive
  - ▶ Calculate E^{FV} of moving two-body systems in a box

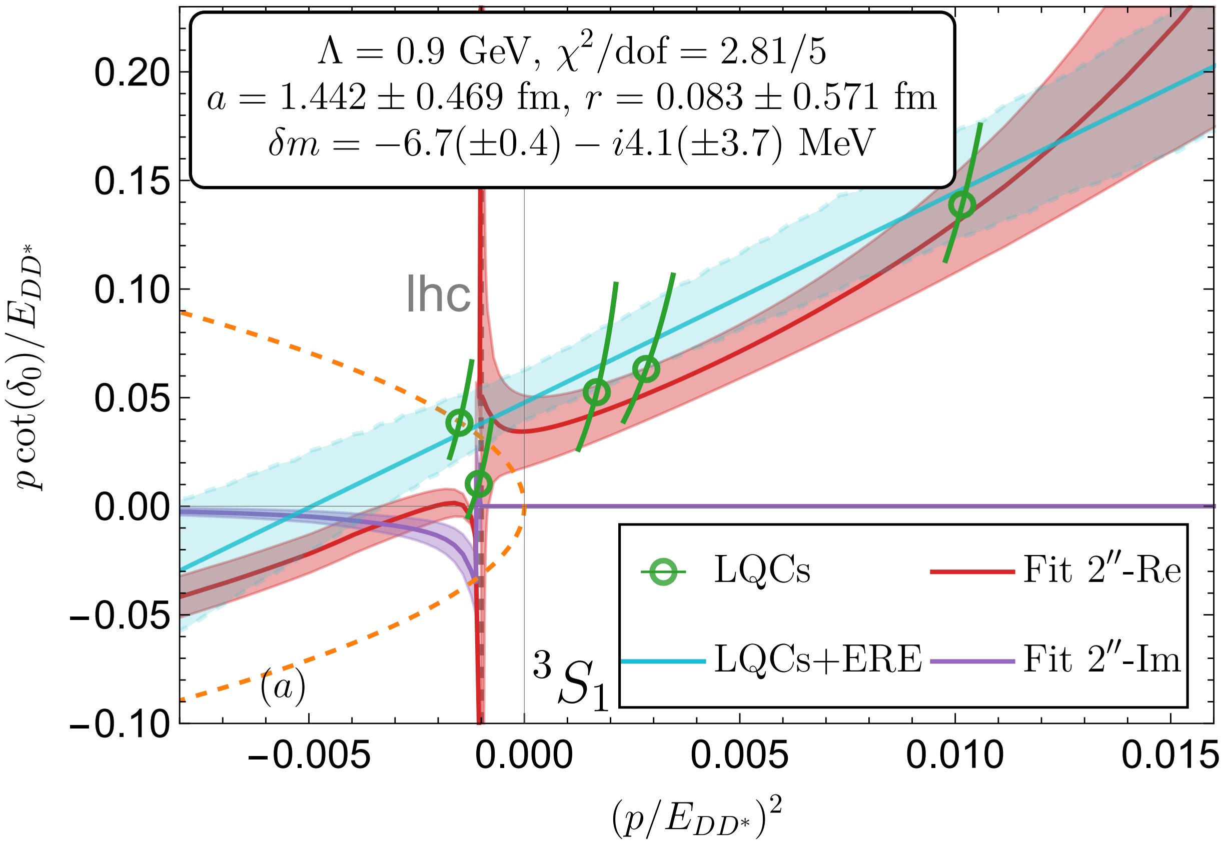
Rummukainen:1995vs,Leskovec:2012gb

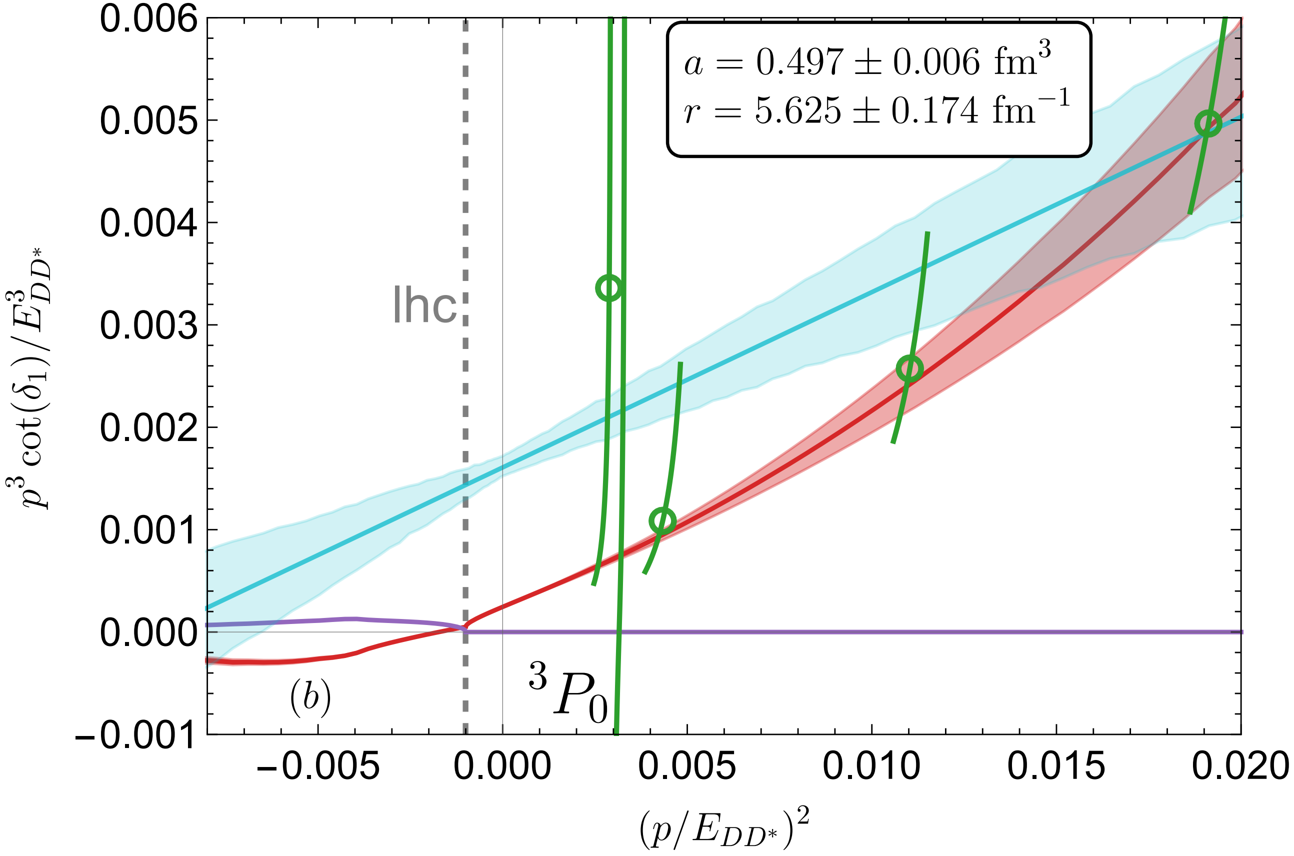
- Box frame (BF) $\boldsymbol{p}$ and center of mass frame (CMF) $\boldsymbol{p}^*$
  - ▶ BF: $\boldsymbol{p} = \frac{2\pi}{L}\boldsymbol{n};$ CMF: $\boldsymbol{p}^* = \gamma^{-1}\left(\boldsymbol{p}_\parallel - \frac{A}{2}\boldsymbol{P}\right) + \boldsymbol{p}_\perp$
  - ▶ For moving systems with $m_1 \neq m_2$, states with different parities could mix
- $\boldsymbol{d} = (0,0,1)$, D_{4h} group for $m_1 = m_2$, C_{4v} group for $m_1 \neq m_2$
- $\boldsymbol{d} = (1,1,0)$, …

Including SD transition terms



Including 3P2 term





Lüscher's formula

- Lippmann-Schwinger equation in the finite volume Luscher:1990ux,Polejaeva:2012ut

$$T^L(\boldsymbol{p},\boldsymbol{q};z) = V(\boldsymbol{p},\boldsymbol{q}) + \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} V(\boldsymbol{p},\boldsymbol{q}) G_0^L(\boldsymbol{k};z) T(\boldsymbol{k};z)$$

$$G_0^L(\boldsymbol{k},z) = (\frac{2\pi}{L})^3 \sum_{\boldsymbol{p}\in\frac{2\pi}{L}\boldsymbol{n}} \frac{2\mu\delta^3(\boldsymbol{p}-\boldsymbol{k})}{q_0^2-\boldsymbol{p}^2} = \mathrm{P.V.}\frac{2\mu}{q_0^2-\boldsymbol{k}^2} + G_F(\boldsymbol{k},z) = G_K(\boldsymbol{k},z) + G_F(\boldsymbol{k},z)$$

with $z = m_1 + m_2 + \frac{q_0^2}{2\mu}$

- The "=" relation is valid up to the exponentially suppressed terms in L
- K matrix in the infinite volume: $K = V + VG_K K$

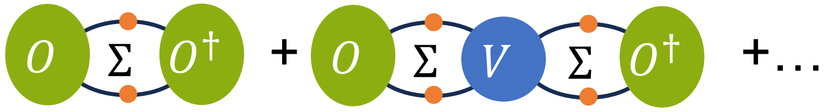
$$T^L = V + V(G_K + G_F)T^L = K + KG_F T^L$$

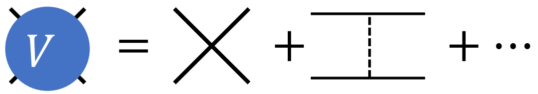
- E^{FV} corresponding to poles of T^L : interaction-independent form

$$\det[1 - KG_F] = 0, \text{ or } \det[G_F - K^{-1}] = 0$$

Detailed derivation of Lüscher's formula

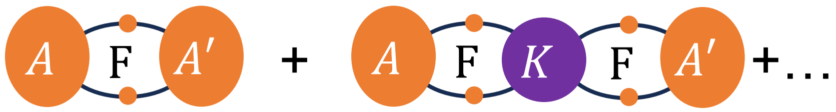
$C_L(P) =$





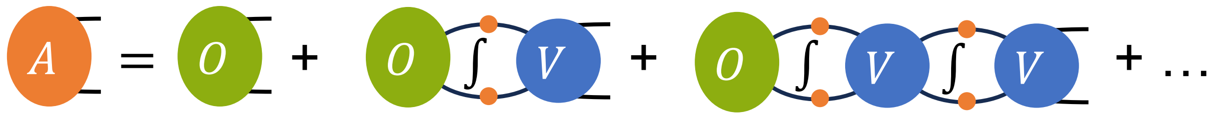
$$\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\right] f(k) = \begin{cases} \mathcal{O}(e^{-mL}) & \text{smooth } f(k) \\ \text{power of } L & \text{otherwise} \end{cases}$$

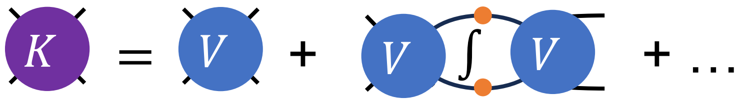
$C_L(P) = C_\infty(P) +$



Within on-shell approximation:

$$F + FKF + \cdots = F(1 - KF)^{-1} = (F^{-1} - K)$$





Note: all the $\int$ should be treated in the sense of P.V.

Lüscher's formula: partial wave mixing effect

- Expanding it in partial wave basis

$$\det[G_F - K^{-1}] = 0, \Rightarrow \det[M_{l'm',lm} - \delta_{ll'}\delta_{mm'}\cot\delta_l] = 0$$

  - ▶ Determinate equation of a matrix with infinite dimensions.
  - ▶ Truncate at some $l_{\max}$
- Reduce to irreps. Γ_i of point group

$$\det[F_{l'm',lm}] = 0 \Rightarrow \begin{vmatrix} F_{\Gamma_1} & & \\ & F_{\Gamma_2} & \\ & & \ddots \end{vmatrix} = 0, \Rightarrow \det[F_{\Gamma_i}] = 0$$

- Obtain the basis of the irreps. $|l, m\rangle \to |\Gamma, l, \alpha\rangle$ (Projection operator technique) Bernard:2008ax
- Lüscher quantization conditions: $\det\left[M^{(\Gamma,\boldsymbol{P})}_{ln,l'n'} - \delta_{ll'}\delta_{nn'}\cot\delta_l\right] = 0$

Luscher:1990ux,Rummukainen:1995vs,Feng:2004ua,Kim:2005gf,Fu:2011xz,Polejaeva:2012ut,Leskovec:2012gb,Gockeler:2012yj,...

Lüscher's formula: partial wave mixing effect

- Example $\boldsymbol{d} = (0,0,1)$, $\Gamma = A_1^+$, w_{lm} depends on E but independent on V

$$\det\left[M^{(\Gamma,\boldsymbol{P})}_{ln,l'n'} - \delta_{ll'}\delta_{nn'}\cot\delta_l\right] = 0, \quad M^{(A_1^+,\boldsymbol{d})} = \begin{bmatrix} w_{00} & -\sqrt{5}w_{20} & \cdots \\ -\sqrt{5}w_{20} & w_{00} + \frac{10}{7}w_{20} + \frac{18}{7}w_{40} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

- Truncate at $l_{max} = 0$, one-to-one relation: $\delta_0(E^{FV}) \sim E^{FV}$
- Truncate at $l_{max} > 0$, no one-to-one relation
  - E.g. $\{E_1^{FV}, E_2^{FV}\} \neq \{\delta_S(E_1^{FV}), \delta_S(E_2^{FV}), \delta_D(E_1^{FV}), \delta_D(E_2^{FV})\}$
  - One has to parameterize the K-matrix
  - Effective range expansion
- Lüscher's formula: quantization conditions in partial wave basis
- Why not quantization conditions in plane wave basis + Hamiltonian method?

Hamiltonian approach in Plane wave basis: $|p_n, \eta\rangle$

- Seven patterns of representation space $\{n_1, n_2, n_3\}_{dim}$ for O_h group

  $\Rightarrow \{0,0,0\}_{1\times 3}, \{0,0,a\}_{6\times 3}, \{0,a,a\}_{12\times 3}, \{0,a,b\}_{24\times 3}\ldots$

- Reduce to irreducible representations (irreps): projection operator

e.g. textbook by M.Dresselhaus et.al

$$\hat{P}^{\Gamma_a}_{\alpha\beta} \equiv \sum_{g_i \in G} \frac{N(\Gamma_a)}{n_G} R^{\Gamma_a}_{\alpha\beta}(g_i)^* \hat{D}(g_i), \quad \hat{P}^{\Gamma_a}_{\alpha\alpha'}|\psi\rangle = a^{\Gamma_a}_{\alpha'}|\Gamma_a, \alpha\rangle.$$

- An example: $\{0,0,a\}_{6\times 3} = 2T_1^+ \oplus T_2^+ \oplus A_1^- \oplus E_1^- \oplus T_1^- \oplus T_2^-$

- For moving systems, elongated boxes, particles with arbitrary spin...

Symmetric group (character table) $\xrightarrow{\hat{P}^{\Gamma}}$ unitary irrep matrices $\xrightarrow{\hat{P}^{\Gamma}_{\alpha\beta}}$ rep space $|p_n\rangle \rightarrow$ irreps

- dim of the $\mathbb{H}_\Gamma$: cubic function of L^{-1}

$$\dim \sim \left(\frac{\Lambda_{\text{UV}}}{2\pi/L}\right)^3 \times \frac{1}{10} \sim \mathcal{O}(1000)$$

Test consistency with Lüscher's formula

- Using the LECs of our best fitting
- V_{full} and S-wave-projected V_S
- Obtain IFV T-matrix

VS

Physical S-D mixing effect

- Obtain the FV energy levels

E^{FV} VS : S-D mixing, S-P mixing

- Using Lüscher's QCs to get δ
  - ▶ Single partial wave QCs

1PW QCs fails

QCs and FVEFT Consistency

QCs and FVEFT inconsistency: LHC